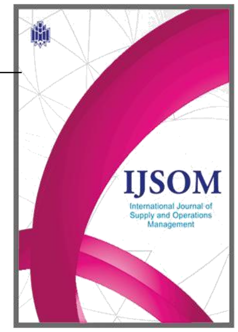
International Journal of Supply and Operations Management

IJSOM

August 2017, Volume 4, Issue 3, pp. 202-214
ISSN-Print: 2383-1359
ISSN-Online: 2383-2525
www.ijsom.com



Cooperative Grey Game: Grey Solutions and an Optimization Algorithm

Osman Palanci[a], Mehmet Onur Olgun[a], Serap Ergun[a], Sırma Zeynep Alparslan Gok[a,*] and Gerhard Wilhelm Weber[b]

[a] *Süleyman Demirel University, Isparta, Turkey*
[b] *Institute of Applied Mathematics, Middle East Technical University and Poznan University of Technology, Poznan, Poland*

Abstract
In this paper, some set-valued solutions using grey payoffs, namely, the grey core, the grey dominance core, and the grey stable sets of cooperative grey games are introduced and studied. The findings of the study demonstrated the relations between the grey core, the grey dominance core, and stable sets of grey cooperative games. Moreover, we present a linear programming (LP) problem for the grey core. We also suggest a corresponding optimization-based algorithm that demonstrates the grey core element of a cooperative grey game. Finally, we introduce an application of how cooperative grey game theory can be used to model users' behaviors in various multimedia social networks. The paper ends with a conclusion and an outlook for future investigations.

Keywords: Cooperative grey games; Grey core; Grey dominance core; Grey stable sets; Linear optimization; Social networks.

1. Introduction

Online community studies in social networks have gained a significant role recently because of the popularity of online social media (Li, Pan, Xiao and Huang, 2014). Social network analysis Scott (2012) encloses a well-developed suite of measures and metrics based on graph theory, spectural theory, and optimization theory which are useful to measure the structural and statistical properties of social networks. Nevertheless, the current social network analysis techniques are insufficient for some reasons. For example, they do not capture the behavior of the individuals in social networks in a convincing way. Also, they do not capture the dynamics of strategic interaction among individuals clearly (Narahari and Narayanam, 2011a).

Game theory can be a natural tool to overcome these fundemental problems, as it provides a rich suite of mathematical models of interactions between individuals (players). Game theoretic models are collateral to the current social network analysis techniques and add a new viewpoint to the area of social network analysis (Chen et al., 2010; Goel and Ronaghi, 2012; Kleinberg and Tardos, 2008; Narahari and Narayanam, 2011b,c; Roy et al., 2016; Terlutter and Capella, 2013).

Cooperative game theory has been studied in various applications, where decision making normally involves a set of

Corresponding author email address: zeynepalparslan@yahoo.com

players, for instance, the paper Saad et al. (2009) applied cooperative game theory to the analysis of communications

in wireless networks. Instead, Bockarjova et al. (2010) studied cooperative game to energy supply system planning; Gilles (2010) investigated applications of cooperative game theory for the analysis of the centrality and power of social networks; and Liu, Li and Yue (2007) proposed a method for discovering different groups from given objects based on cooperative game theory (Tijs, 2003).

For this reason, a closer analysis of the adoption of social networks from an interdisciplinary perspective is needed. This is where our approach comes into play. In our paper, we focus on some set-values solutions using grey payoffs while paying attention to the grey core element of a cooperative grey game. More specifically, we discuss how cooperation can be important in social network analysis systems formed by cooperative grey games. In this study, we model our problems in the framework of cooperative game theory.

Grey system theory is one of the new mathematical theories born out of the concept of the grey numbers. A grey system is defined as a system containing information presented as grey numbers; and a grey decision is defined as a decision made within a grey system (Deng, 1982, 1985; Zhang et al., 2005).

As a matter of fact, many researchers have handled this vagueness with the help of the grey numbers, one of the keystones in grey system theory. When decision information is given by a grey number, games are called cooperative grey games (Fang and Liu, 2003; Kose and Forest, 2015). There are many applications in cooperative game theory with grey uncertainty (Ekici et al., 2018; Alparslan Gok et al., 2018).

Our main contribution is to propose a cooperative game-theoretic model under grey data in social network analysis problems, and to discuss how grey numbers compared to other concepts of uncertain information. Based on this approach, we study an algorithm for finding a grey core element of cooperative grey game. We suggest an Linear Programming (LP) approach of the grey core inspired by using Bondereva-Shapley characterization. Extensive application of three social network analysis companies is considered. Using our algorithm, we obtain a grey core element of the grey cost game related to social network companies. The application results demonstrate a high effectiveness of our method.

The remainder of this paper is structured as follows. We give basic notions and facts from the theory of grey calculus in Section 2. In Section 3, we introduce the grey solutions, i.e., the grey imputation set and the grey core. We study relations between the grey core, the grey dominance core, and the grey stable sets for cooperative grey games in Section 4. In Section 5, we present a new LP-based algorithm about finding the grey core element of the cooperative grey game and a pseudo code of the algorithm. A real-life application is given in Section 6. Finally, we conclude this paper, presenting also an outlook to future research challenges.

2. Preliminaries

A cooperative game in coalitional form is an ordered pair $< N, v >$, where $N = \{1,\ldots,n\}$ is the set of players and $v : 2^N \to \mathbb{R}$ is a map, assigning to each coalition $S \in 2^N$ a real number, such that $v(\varnothing) = 0$. This function v is called the characteristic function of the game and $v(S)$ is called the worth (or value) of coalition S. Often we identify a game $< N, v >$ with its characteristic function v. The set of coalitional games with player set N is denoted by G^N (for details see Zhang et al., 2005).

Now, we give some preliminaries from grey calculus and use it for cooperative grey games (Liu and Lin, 2006; Palanci et al., 2015). A grey number is such a number whose exact value is not known but a range within that value lies is known. In applications, a grey number is an interval or a general set of numbers. In this paper, we consider the interval grey numbers.

A grey number with both a lower limit $\underline{\Gamma}$ and an upper limit $\overline{\Gamma}$ is called an interval grey number, denoted as $\Gamma \in [\underline{\Gamma}, \overline{\Gamma}]$. For example, the weight of a seal is between 20 and 25 kg. A specific person's height is between 1.8

and 1.9 meters. A reaction time of a group of humans to some advertisement stimulus is between 3 to 20 seconds. These grey numbers can be respectively written as

$\Gamma_1 \in [20,25], \Gamma_2 \in [1.8,1.9]$ and $\Gamma_3 \in [3,20].$

Now, we discuss various operations on interval grey numbers. Let

$\Gamma_1 \in [\underline{\Gamma}_1, \overline{\Gamma}_1], \underline{\Gamma}_1 < \overline{\Gamma}_1,$ and $\Gamma_2 \in [\underline{\Gamma}_2, \overline{\Gamma}_2], \underline{\Gamma}_2 < \overline{\Gamma}_2.$

The sum of Γ_1 and Γ_2, written $\Gamma_1 + \Gamma_2$, is defined by

$$\Gamma_1 + \Gamma_2 \in [\underline{\Gamma}_1 + \underline{\Gamma}_2, \overline{\Gamma}_1 + \overline{\Gamma}_2]$$

For example, let $\Gamma_1 \in [3,4]$ and $\Gamma_2 \in [5,8]$, then, $\Gamma_1 + \Gamma_2 \in [8,12]$. Assume that $\Gamma \in [\underline{\Gamma}, \overline{\Gamma}], \underline{\Gamma} < \overline{\Gamma},$ and k is a positive real number. The scalar multiplication of k and Γ is defined by $k\Gamma \in [k\underline{\Gamma}, k\overline{\Gamma}]$.

We denote by $\Gamma(\mathrm{P})$ the set of interval grey numbers in P. Let $\Gamma_1, \Gamma_2 \in \Gamma(\mathrm{P})$ with $[\underline{\Gamma}_1, \overline{\Gamma}_1], \underline{\Gamma}_1 < \overline{\Gamma}_1$; $\Gamma_2 \in [\underline{\Gamma}_2, \overline{\Gamma}_2], \underline{\Gamma}_2 < \overline{\Gamma}_2$, $|\Gamma_1| = \overline{\Gamma}_1 - \underline{\Gamma}_1$ and $\alpha \in \mathrm{P}_+$. Then,

(i) $\Gamma_1 + \Gamma_2 \in [\underline{\Gamma}_1 + \underline{\Gamma}_2, \overline{\Gamma}_1 + \overline{\Gamma}_2]$;

(ii) $\alpha\Gamma \in [\alpha\underline{\Gamma}, \alpha\overline{\Gamma}]$.

By (i) and (ii) we see that $\Gamma(\mathrm{P})$ has a *cone* structure. In general, the difference of Γ_1 and Γ_2 is defined as follows (see Moore, 1979):

$$\Gamma_1 ! \ \Gamma_2 = \Gamma_1 + (-\Gamma_2) \in [\underline{\Gamma}_1 - \overline{\Gamma}_2, \overline{\Gamma}_1 - \underline{\Gamma}_2]$$

For example, let $\Gamma_1 \in [6,8]$, and $\Gamma_2 \in [2,5]$, then we have

$\Gamma_1 ! \ \Gamma_2 \in [6-5, 8-2] = [1,6],$

$\Gamma_2 ! \ \Gamma_1 \in [2-8, 5-6] = [-6,-1].$

Differently from the subtraction operator introduced by Moore (1979), we introduce a new grey subtraction operator. In fact, $\Gamma_1 - \Gamma_2$ is defined, only if $|\overline{\Gamma}_1 - \underline{\Gamma}_1| \geq |\overline{\Gamma}_2 - \underline{\Gamma}_2|$, by $\Gamma_1 - \Gamma_2 \in [\underline{\Gamma}_1 - \underline{\Gamma}_2, \overline{\Gamma}_1 - \overline{\Gamma}_2]$. Note that $\underline{\Gamma}_1 - \underline{\Gamma}_2 \leq \overline{\Gamma}_1 - \overline{\Gamma}_2$. We say that Γ_1 is grey weakly better than Γ_2, which we denote by $\Gamma_1 \neq \Gamma_2$, if and only if $\underline{\Gamma}_1 \geq \underline{\Gamma}_2$ and $\overline{\Gamma}_1 \geq \overline{\Gamma}_2$. We also use the reverse notation $\Gamma_1 \div \Gamma_2$, if and only if $\underline{\Gamma}_1 \leq \underline{\Gamma}_2$ and $\overline{\Gamma}_1 \leq \overline{\Gamma}_2$.

Moreover, we want to introduce a new equality operator. Then, we say that Γ_1 is *grey equal* to Γ_2, which we denote by $\Gamma_1 \approx \Gamma_2$, if and only if $\underline{\Gamma}_1 = \underline{\Gamma}_2$ and $\overline{\Gamma}_1 = \overline{\Gamma}_2$. We note that $\Gamma_1 \not\approx \Gamma_2$, if and only if $\underline{\Gamma}_1 \neq \underline{\Gamma}_2$ and $\overline{\Gamma}_1 \neq \overline{\Gamma}_2$. We say that Γ_1 is *grey better* than Γ_2, which we denote by $\Gamma_1 \succ \Gamma_2$, if and only if $\Gamma_1 \neq \Gamma_2$ and $\Gamma_1 \not\approx \Gamma_2$. We also use the reverse notation $\Gamma_1 \prec \Gamma_2$, if and only if $\Gamma_1 \div \Gamma_2$ and $\Gamma_1 \not\approx \Gamma_2$.

Let $\Gamma_1 \in [2,5]$, and $\Gamma_2 \in [6,8], \Gamma_1 - \Gamma_2$ is defined since $|5-2| \geq |8-6|$, but $\Gamma_2 - \Gamma_1$ is not defined since $|8-6| = 23 = |5-2|$; then we have

$\Gamma_1 - \Gamma_2 \in [2-6, 5-8] = [-4,-3].$

We recall that a cooperative grey game is an ordered pair $< N, w' >$, where $N = \{1, \ldots, n\}$ is the set of players, and $w' : 2^N \rightarrow \Gamma(\mathrm{P})$ is the characteristic function such that $\Gamma_{\varnothing} = w'(\varnothing) \in 0,0]$. The grey payoff function value $w'(S) \in [\underline{w'(S)}, \overline{w'(S)}]$ refers to the value of the grey expectation benefit belonging to a coalition $S \in 2^N$, where $\underline{w'(S)}$ and $\overline{w'(S)}$ represent the maximum and minimum possible profits of the coalition S. So, a cooperative grey game can be considered as a classical cooperative game with grey profits Γ_i. Grey solutions are useful to solve reward or cost sharing problems with grey data, using cooperative grey games as a tool. The building blocks of grey solutions are grey payoff vectors, i.e., vectors whose components belong to $\Gamma(\mathrm{P})$. We denote by $\Gamma(\mathrm{P})^N$ the set of all such grey payoff vectors, and we designate by ΓG^N the family of all cooperative grey games.

Remark 1 *The difference between cooperative interval games and cooperative grey games is as follows: In cooperative interval games, the coalition values are closed and bounded intervals in* P *, but in cooperative grey games, the coalitional values are real numbers chosen from a real interval.*

3. The grey core

In this section, we introduce a new solution for cooperative grey games. These are grey imputation set and the grey core.

The *grey imputation set* $\mathrm{I}\,(w')$ of the grey game w' is defined by

$$\mathrm{I}\,(w') := \left\{ (\Gamma_1, \ldots, \Gamma_n) \in \Gamma(\mathrm{P})^N \mid \sum_{i \in N} \Gamma_i \approx w'(N), w'(\{i\}) \div \Gamma_i \text{for all } i \in N \right\}. \tag{1}$$

We note that $\sum_{i \in N} \Gamma_i \approx w'(N)$ is equivalent with $\sum_{i \in N} \underline{\Gamma_i} = \underline{w'(N)}$ and $\sum_{i \in N} \overline{\Gamma_i} = \overline{w'(N)}$, and $w'(\{i\}) \div \Gamma_i$ is equivalent with $\underline{w'(\{i\})} \leq \underline{\Gamma_i}$ and $\overline{w'(\{i\})} \leq \overline{\Gamma_i}$.

Notice that the grey uncertainty of coalition values propagates into the grey uncertainty of individual payoffs, and we obtain grey payoff vectors as building blocks of grey solutions. The grey imputation set consists of those grey payoff vectors which assure the distribution of the grey uncertain worth of the grand coalition such that each player can expect a grey weakly better grey payoff than what he/she can expect on his/her own.

The *grey core* $\mathrm{X}(w')$ of the grey game w' is defined by

$$\mathrm{X}(w') := \left\{ (\Gamma_1, \ldots, \Gamma_n) \in \Gamma(\mathrm{P})^N \mid \sum_{i \in N} \Gamma_i \approx w'(N), \sum_{i \in S} \Gamma_i \neq w'(S), \forall S \in 2^N \setminus \{\varnothing\} \right\}. \tag{2}$$

Here, $\sum_{i \in N} \Gamma_i \approx w'(N)$ is the grey efficiency condition and $\sum_{i \in S} \Gamma_i \neq w'(S)$, $S \in 2^N \setminus \{\varnothing\}$, are the grey stability conditions of the grey payoff vectors. Clearly, $\mathrm{X}(w') \subseteq \mathrm{I}\,(w')$.

The following example illustrates computing of the grey core for a cooperative grey game.

Example 2 *Consider the cooperative grey game, where* $N = \{1,2,3\}$ *is the set of players and the grey coalitional values are* $w'(13) \in [7,8]$, $w'(12) \in [12,17]$, $w'(N) \in [24,29]$, *and* $w'(S) \in [0,0]$ *in any other case. Now, we want to find the grey core of the cooperative grey game. From the definition of the grey core, it follows that:*

$\mathrm{X}(w') = \{(\Gamma_1, \ldots, \Gamma_n) \in \Gamma(\mathrm{P})^N \mid$

$\underline{\Gamma_1} + \underline{\Gamma_2} + \underline{\Gamma_3} = 24$ and $\overline{\Gamma_1} + \overline{\Gamma_2} + \overline{\Gamma_3} = 29$,

$\underline{\Gamma_1} + \underline{\Gamma_2} \geq 12$ and $\overline{\Gamma_1} + \overline{\Gamma_2} \geq 17$,

$\underline{\Gamma}_1 + \underline{\Gamma}_3 \geq 7$ and $\overline{\Gamma}_1 + \overline{\Gamma}_3 \geq 8$,

$\underline{\Gamma}_2 + \underline{\Gamma}_3 \geq 0$ and $\overline{\Gamma}_2 + \overline{\Gamma}_3 \geq 0$,

$\underline{\Gamma}_1, \underline{\Gamma}_2, \underline{\Gamma}_3, \overline{\Gamma}_1, \overline{\Gamma}_2, \overline{\Gamma}_3 \geq 0\}$.

One can easily check the following for the grey allocation: while the grey payoff vector $\Gamma \in ([5,8],[7,8],[12,13])$ do not belong to the grey core of the cooperative grey game, i.e., $\Gamma \notin X(w')$; the grey payoff vector $H \in ([5,8],[7,9],[12,12])$ belongs to the grey core of the cooperative grey game, i.e., $H \in X(w')$.

A grey game $w' \in \Gamma G^N$ is called Γ-*balanced* if for each balanced map $\lambda : 2^N \setminus \{\varnothing\} \to P_+$ we have $\sum_{S \in 2^N \setminus \{\varnothing\}} \lambda(S) w'(S) \preccurlyeq w'(N)$. In classical cooperative game theory, they independently give a characterization of games with a non-empty core (Bondareva, 1963; Shapley, 1967). By Theorem 3, we extend this result to cooperative grey games.

Theorem 3 *Let* $w' \in \Gamma G^N$. *Then, the game* w' *is* Γ*-balanced if and only if* $X(w')$ *is nonempty.*

4. The grey dominance core and grey stable sets

In this section, we introduce the definition of the grey dominance core and the grey stable set. Then, we study relations between the grey core, the grey dominance core and the grey stable sets.

Definition 4 *Let* $< N, w' >$ *be an* n*-person grey game. Let* $\Gamma = (\Gamma_1, \ldots, \Gamma_n)$, $H = (H_1, \ldots, H_n) \in I(w')$ and $S \in 2^N \setminus \{\varnothing\}$. Then Γ *dominates* H via coalition S, and we designate this by Γ dom_S H, if

(i) $\Gamma_i \neq H_i$ for all $i \in S$

(ii) $\sum_{i \in S} \Gamma_i \preccurlyeq w'(S) \Leftrightarrow \sum_{i \in S} \underline{\Gamma}_i \leq \underline{w'(S)}$ and $\sum_{i \in S} \overline{\Gamma}_i \leq \overline{w'(S)}$.

For $S \in 2^N \setminus \{\varnothing\}$, we denote by $\Delta(S)$ the set of those elements of $I(w')$ which are dominated via S. For $\Gamma, H \in I(w')$, we say that Γ *dominates* H and denote it by Γ dom H if there is an $S \in 2^N \setminus \{\varnothing\}$ such that Γ dom_S H. Furthermore, Γ is called *undominated* if there exist no H and no coalition S such that H dom_S Γ.

Definition 5 *The grey dominance core* $\Delta X(w')$ *of a grey game* $w' \in \Gamma G^N$ *is the set*

$$\Delta X(w') := I(w') \setminus \sum_{S \in 2^N \setminus \{\varnothing\}} \Delta(S)$$

i.e., it consists of all undominated elements in $I(w')$.

In the next theorem, we state that $X(w') \subseteq \Delta X(w')$.

Theorem 6 *The grey core is a subset of the grey dominance core for each grey game.*

Proof. Let $< N, w' >$ be a grey game and $H \notin \Delta X(w')$. Then there is a $\Gamma \in I(w')$ and a coalition S such that Γ dom_S H. Then $w'(S) \succcurlyeq \sum_{i \in S} \Gamma_i \succ \sum_{i \in S} H_i$, which implies that $H \notin X(w')$.

Now, we introduce the definition of a superadditive grey game which is necessary for Theorem 7. A grey game $w^{'} \in \Gamma G^{N}$ is said to be **superadditive** if for all $S,T \subset N$ with $S \cap T = \varnothing$ the following two conditions hold:

i) $w^{'}(S \cup T) \succeq w^{'}(S) + w^{'}(T)$,

ii) $\left|w^{'}\right|(S \cup T) \geq \left|w^{'}\right|(S) + \left|w^{'}\right|(T)$.

For superadditive games the grey core and the grey dominance core coincide as the following theorem shows.

Theorem 7 *Let* $< N, w^{'} >$ *be a superadditive game. Then* $\Delta X(w^{'}) = X(w^{'})$.

Proof. (i) Firstly, we show that for an $H \in I(w^{'})$ with $\sum_{i \in S} H_i \prec w^{'}(S)$ for some S, there is an $\Gamma \in I(w^{'})$ such that Γ dom$_S$ H. Let us define Γ as follows. If $i \in S$, herewith $\Gamma_i :\approx H_i + |S|^{-1}(w^{'}(S) - \sum_{i \in S} H_i)$. If $i \notin S$, then $\Gamma_i :\approx w^{'}(\{i\}) + (w^{'}(N) - w^{'}(S) - \sum_{i \in N \setminus S} w^{'}(i))|N \setminus S|^{-1}$. Now, $\Gamma \in I(w^{'})$, where for the proof of $\Gamma_i \succeq w^{'}(\{i\})$ for $i \in N \setminus S$, we use the superadditivity of the game. Furthermore, Γ dom$_S$ H. (ii) To prove $\Delta X(w^{'}) = X(w^{'})$, we have, in view of Theorem 6, only to show that $\Delta X(w^{'}) \subset X(w^{'})$. Suppose $H \in \Delta X(w^{'})$. Then, there is no $\Gamma \in I(w^{'})$ with Γ dom H. In view of (i), we have $\sum_{i \in S} H_i \succeq w^{'}(S)$ for all $S \in 2^N \setminus \{\varnothing\}$. Hence, $H \in X(w^{'})$.

The inclusions stated in the previous theorem may be rather strict. The following example illustrates that the inclusion of $X(w^{'})$ in $\Delta X(w^{'})$ might be strict indeed.

Example 8 *Let* $< N, w^{'} >$ *be the three-person grey game with* $w^{'}(12) \in [2,2]$, $w^{'}(N) \in [1,1]$ *and* $w^{'}(S) \in [0,0]$ *if* $S \neq \{\{1,2\}, N\}$. *Then,* $X(w^{'}) = \varnothing$ *because the game is not* Γ*-balanced. We note that*

$$w^{'}(12) + w^{'}(3) \succ w^{'}(N) \Leftrightarrow (\underline{w^{'}(12)} + \underline{w^{'}(3)} = 2 + 1 = 3 > 1 = \underline{w^{'}(N)}$$
$$\overline{w^{'}(12)} + \overline{w^{'}(3)} = 2 + 1 = 3 > 1 = \overline{w^{'}(N)}).$$

Further, $D(S) = \varnothing$ if $S \neq \{1,2\}$ and $D(\{1,2\}) = \{\Gamma \in I(w^{'}) \mid \Gamma_3 \succ \Gamma_{\varnothing}\}$. The elements $\Gamma \in I(w^{'})$ which are undominated satisfy $\Gamma_3 \approx \Gamma_{\varnothing}$. Since the grey dominance core is the set of undominated elements in $I(w^{'})$, the grey dominance core of this games is nonempty.

Definition 9 *Let* $< N, w^{'} >$ *be a grey game and* $A \subseteq I(w^{'})$. *The set* A *is called a grey stable set if the following conditions hold* [1]*:*

(i) (Internal stability) There do not exist $\Gamma, H \in A$ *such that* Γ *dom* H.

(ii) (External stability) For each $\Gamma \notin A$ *there exists* $H \in A$ *such that* H dom Γ.

In the following theorem, we show relations between the grey core, the grey dominance core, and the grey stable sets for cooperative grey games.

Theorem 10 *Let* $w^{'} \in \Gamma G^{N}$ *and let* A *be a stable set of* $w^{'}$. *Then,* $X(w^{'}) \subseteq \Delta X(w^{'}) \subseteq A$.

[1] The classical notion of stable sets was introduced in von Neumann and Morgenstern (1944).

Proof. We know that $\mathrm{X}(w') \subset \Delta\mathrm{X}(w')$ from Theorem 6 to prove $\Delta\mathrm{X}(w') \subset A$ it is sufficient to show $(\mathrm{I}\,(w') \setminus A) \subset (\mathrm{I}\,(w') \setminus \Delta\mathrm{X}(w'))$. Let us take $\Gamma \in \mathrm{I}\,(w') \setminus A$. Given the external stability of A, there is a $\mathrm{H} \in A$ with H dom Γ. Since the elements in $\Delta\mathrm{X}(w')$ are not dominated, we obtain $\Gamma \notin \Delta\mathrm{X}(w')$ or, equivalently, $\Gamma \in \mathrm{I}\,(w') \setminus \Delta\mathrm{X}(w')$. The proof is completed.

Now, we introduce the unanimity grey games and prove that the grey core and the grey dominance core coincide on the class of unanimity grey games. Let $\mathrm{H} \in \Gamma(\mathrm{P}_+)$ and let $T \in 2^N \setminus \{\varnothing\}$. The unanimity grey game based on H and T is defined by

$$u_{T,\mathrm{H}}(S) = \begin{cases} \mathrm{H}, & T \subseteq S, \\ \Gamma_\varnothing, & \text{otherwise}, \end{cases} \tag{3}$$

for each $S \in 2^N$. We recall that for all $T \in 2^N \setminus \{\varnothing\}$, the classical unamity game based on T, $< N, u_T >$, is defined by

$$u_T(S) = \begin{cases} 1, & T \subseteq S, \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

for each $S \in 2^N \setminus \{\varnothing\}$.

The core $C(u_T)$ of the unanimity game $< N, u_T >$ is given by $C(u_T) = \{x \in \mathrm{P}^N \mid \sum_{i=1}^n x_i = 1,$ and $x_i = 0$ for $i \in N \setminus T\}$.

The next proposition provides a description of the grey core of an unanimity grey game and shows that on the class of unanimity games the grey core and the grey dominance core coincide. We define K as follows:

$$K := \{(\Gamma_1, \ldots, \Gamma_n) \in \Gamma(\mathrm{P})^N \mid \sum_{i\in N} \Gamma_i \approx \mathrm{H}, \underline{\Gamma_i} \geq 0 \text{ for all } i \in N, \Gamma_i \approx \Gamma_\varnothing \text{ for } i \in N \setminus T\}.$$

Proposition 11 *Let $< N, u_{T,\mathrm{H}} >$ be the unanimity grey game based on coalition T and the grey payoff $\mathrm{H} \neq \Gamma_\varnothing$. Then,* $\Delta\mathrm{X}(u_{T,\mathrm{H}}) = \mathrm{X}(u_{T,\mathrm{H}}) = K$.

Proof. Firstly, we prove that the grey core of $u_{T,\mathrm{H}}$ can be described as the set K. In order to show that $\mathrm{X}(u_{T,\mathrm{H}}) \subset K$, let $(\Gamma_1, \ldots, \Gamma_n) \in \mathrm{X}(u_{T,\mathrm{H}})$. Clearly, for each $i \in N$ we have $\Gamma_i \neq u_{T,\mathrm{H}}(\{i\})$ and $u_{T,\mathrm{H}}(\{i\}) \neq \Gamma_\varnothing$. So $\underline{\Gamma_i} \geq 0$ for all $i \in N$. Furthermore, $\sum_{i\in N} \Gamma_i \approx u_{T,\mathrm{H}}(N) \approx \mathrm{H}$. Since also $\sum_{i\in T} \Gamma_i \neq \mathrm{H}$, we conclude that $\Gamma_i \approx \Gamma_\varnothing$ for $i \in N \setminus T$. So, $(\Gamma_1, \ldots, \Gamma_n) \in K$. In order to show that $K \subset \mathrm{X}(u_{T,\mathrm{H}})$, let $(\Gamma_1, \ldots, \Gamma_n) \in K$. So, $\underline{\Gamma_i} \geq 0$ for all $i \in N$, $\Gamma_i \approx \Gamma_\varnothing$ if $i \in N \setminus T$, $\sum_{i\in N} \Gamma_i \approx \mathrm{H}$. Then $(\Gamma_1, \ldots, \Gamma_n) \in \mathrm{X}(u_{T,\mathrm{H}})$, because it also holds:

(i) $\sum_{i\in S} \Gamma_i \neq \Gamma_\varnothing \approx u_{T,\mathrm{H}}(S)$ if $T \setminus S \neq \varnothing$,

(ii) $\sum_{i\in S} \Gamma_i \approx \sum_{i\in N} \Gamma_i \approx u_{T,\mathrm{H}}(N) \approx \mathrm{H} \approx u_{T,\mathrm{H}}(S)$ if $T \subset S$.

Next, we prove that $\mathrm{X}(u_{T,\mathrm{H}}) = \Delta\mathrm{X}(u_{T,\mathrm{H}})$. We know that $\mathrm{X}(u_{T,\mathrm{H}}) \subset \Delta\mathrm{X}(u_{T,\mathrm{H}})$ by Theorem 6. We only have to demonstrate that $\Delta\mathrm{X}(u_{T,\mathrm{H}}) \subset \mathrm{X}(u_{T,\mathrm{H}})$, or we need to show that for each $\Gamma \notin \mathrm{X}(u_{T,\mathrm{H}})$. Then, there is an $\alpha \in N \setminus T$ with $\Gamma_\alpha \not\approx \Gamma_\varnothing$. Now, $\Gamma' dom_T \Gamma$, where $\Gamma'_i \approx \Gamma_\varnothing$ for $i \in N \setminus T$ and $\Gamma'_i = \Gamma_i + \frac{1}{|T|}\Gamma_\alpha$ for $i \in T$.

So, $\Gamma \notin \Delta\mathrm{X}(u_{T,\mathrm{H}})$.

The grey core may coincide with the grey dominance core also for games which are not unanimity grey games, as the following example shows.

Example 12 *We reconsider the game* w' *in Example 8. We will show that* $\Delta X(w') = X(w')$. *Take* $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3) \in X(w')$. *Note that if* $\Gamma_1 \not\approx \Gamma_\varnothing$ *then* $(\Gamma_\varnothing, \Gamma_2 + \frac{1}{2}\Gamma_1, \Gamma_3 + \frac{1}{2}\Gamma_1) dom_{\{2,3\}} (\Gamma_1, \Gamma_2, \Gamma_3)$. *So,* $\Gamma \notin \Delta X(w')$. *Similarly, if* $\Gamma_2 \not\approx \Gamma_\varnothing$ *then* $\Gamma \notin \Delta X(w')$. *Hence,* $\Delta X(w') \subseteq \{(\Gamma_\varnothing, \Gamma_\varnothing, H)\} = X(w')$ *by Example 8. On the other hand, we know, in view of Theorem 10, that* $X(w') \subseteq \Delta X(w')$. *So, we conclude that* $\Delta X(w') = X(w')$.

5. The optimization problem and algorithm

In this section, we study an algorithm for finding the grey core element of cooperative grey game. We suggest a LP of the grey core inspired by using Bondereva-Shapley characterization (Bondareva, 1963; Shapley, 1967). We present the LP problem of the grey core as a pessimistic and an optimistic scenario. The LP problem of the pessimistic scenario is as follows:

$$\begin{aligned} &\text{minimize} && \sum_{i \in N} \underline{\Gamma_i} \\ &\text{suchthat} && \sum_{i \in S} \underline{\Gamma_i} \geq \underline{w'(S)}, \forall S \subseteq N, \end{aligned} \tag{5}$$

The LP problem of the optimistic scenario is as follows:

$$\begin{aligned} &\text{minimize} && \sum_{i \in N} \overline{\Gamma_i} \\ &\text{suchthat} && \sum_{i \in S} \overline{\Gamma_i} \geq \overline{w'(S)}, \forall S \subseteq N. \end{aligned} \tag{6}$$

Let $(\Gamma_1^*, \Gamma_2^*, \ldots, \Gamma_n^*)$ be an optimal solution of this LP. Then we know:

$$\sum_{i \in N} \underline{\Gamma_i^*} = \underline{w'(N)}, \text{ and } \sum_{i \in N} \overline{\Gamma_i^*} = \overline{w'(N)}.$$

In particular,

$$\underline{w'(N)} = \underline{\Gamma_1^*} + \ldots + \underline{\Gamma_n^*} \leq \overline{\Gamma_1^*} + \ldots + \overline{\Gamma_n^*} = \overline{w'(N)}.$$

There are four possibilities:

1. $\underline{\Gamma_1^*} + \ldots + \underline{\Gamma_n^*} = \underline{w'(N)}$,
2. $\overline{\Gamma_1^*} + \ldots + \overline{\Gamma_n^*} = \overline{w'(N)}$,
3. $\underline{\Gamma_1^*} + \ldots + \underline{\Gamma_n^*} \neq \underline{w'(N)}$,
4. $\overline{\Gamma_1^*} + \ldots + \overline{\Gamma_n^*} \neq \overline{w'(N)}$.

If the first two possibilities hold together, all solutions of this LP problem constitute the core of $< N, w' >$. In fact, the grey core shall consist precisely of the solutions of this linear program. Furthermore, in other cases, the grey core is empty.

Now, we give an algorithm for finding the grey core element of a cooperative grey game by using the above result from LP. The algorithm is stated as follows:

Algorithm The algorithm of finding an element of the grey core

Input: N : The set of players, $\Gamma_i \in \left[\underline{\Gamma_i}, \overline{\Gamma_i}\right]$: the grey payoff vectors; $\underline{w'(N)}, \overline{w'(N)}, \underline{w'(S)}$ and $\overline{w'(S)}$: the grey coalitional values. We know the equations below:

$$\sum_{i\in N}\Gamma_i \approx w'(N) \Leftrightarrow \sum_{i\in N}\underline{\Gamma_i} = \underline{w'(N)} \text{ and } \sum_{i\in N}\overline{\Gamma_i} = \overline{w'(N)},$$

$$\sum_{i\in S}\Gamma_i \neq \overline{w'(S)} \Leftrightarrow \sum_{i\in S}\underline{\Gamma_i} \geq \underline{w'(S)} \text{ and } \sum_{i\in S}\overline{\Gamma_i} \geq \overline{w'(S)}.$$

Output: $\left(\Gamma_1^*,\ldots,\Gamma_n^*\right)$

1: Determine the number of players $N = \{1,\ldots,n\}$ and all possible coalitions $S \subseteq N$. Solve Eqn. (5) for the pessimistic scenario by using the simplex method of LP. If we obtain its feasible solution, then Γ_i^* is the optimal value of the objective function of Eqn. (5). We can say that $\underline{\Gamma_i^*}$ is an element of the grey core $X(w')$, and we go to Step 2. On the contrary, there is not any feasible solution, so the algorithm stops. To calculate the grey payoff vector of each player for $i = 1,\ldots,n,$ solve the following optimization problem:

$$\begin{aligned} &\text{minimize} && \sum_{i\in N}\underline{\Gamma_i} \\ &\text{suchthat} && \sum_{i\in S}\underline{\Gamma_i} \geq \underline{w'(S)}, \forall S \subseteq N, \end{aligned}$$

2: Repeat Step 1 for the optimistic scenario.

3: Check the lower bound value and upper bound value of Γ_i^*.

If $\sum_{i\in N}\underline{\Gamma_i^*} = \underline{w'(N)}$ and $\sum_{i\in N}\overline{\Gamma_i^*} = \overline{w'(N)}$, then insert Γ_i^* and into $X(w')$ set.

Therefore, $\Gamma_i^* \in X(w')$. Else if $\sum_{i\in N}\underline{\Gamma_i^*} \neq \underline{w'(N)}$ or $\sum_{i\in N}\overline{\Gamma_i^*} \neq \overline{w'(N)}$, then it is understood that the core set is empty and can be defined as $X(w') = \varnothing$.

end if

4: If Γ_n^* is defined, then go to Step 5. Else: calculate for the other players $i = +1$ and go Step 1.

5: An optimal solution of the algorithm is given by $\left(\Gamma_1^*,\ldots,\Gamma_n^*\right)$ which is also the output of the algorithm.

The *pseudo code* of the algorithm is given below.

GreyCoreElement $\left(N, \underline{w'(N)}, \overline{w'(N)}, \underline{w'(S)}, \overline{w'(S)}\right)$

do while (Γ_n^* is defined)

  set $i = 1$

   for $i := n$

    compute the **sum** of the elements in $\underline{\Gamma_i^*}$

    find the min value

    sbj to **sum** $\left(S \leftarrow i, \underline{\Gamma_i^*} \geq \underline{w'(S)}\right)$ for all $\forall S \text{—} N$

    compute the **sum** of the elements in $\overline{\Gamma_i^*}$

    find the **min** value

    sbj to **sum** $\left(S \leftarrow i, \overline{\Gamma_i^*} \geq \overline{w'(S)}\right)$ for all $\forall S \text{—} N$

     ifsum $\left(N \leftarrow i, \underline{\Gamma_i^*}\right) = \underline{w'(N)}$ and **sum** $\left(N \leftarrow i, \overline{\Gamma_i^*}\right) = \overline{w'(N)}$

```
11.          then set Γ*_i in X(w')
12.              else if sum (N ← i, Γ*_i(lower) ≠ w'(N)(lower)) or sum (N ← i, Γ*_i(upper) ≠ w'(N)(upper))
13.                  then return X(w') = ∅, go line 20
14.                      if i ≠ N, i = +1 then go line 4
15.                          else return (Γ*_1,…,Γ*_n)
16.                      end if
17.                  end if
18.          end if
19.      end for
20. end while
```

In this algorithm, firstly the lower bounds are calculated. Each lower bound of the grey core element is added to the core sequence as a first parameter. Then, the upper bounds are calculated, and each upper bound of the grey core element is added to the core sequence as a second parameter. At last, the algorithm yields the the grey core elements. Furthermore, all grey core elements are computed.

Example 13 *Let* $< N, w' >$ *be the three-person grey game with* $w'(\{i\}) \in [0,0]$ *for all* $i \in N$, $w'(12) \in [15,21]$, $w'(13) \in [20,30]$, $w'(23) \in [25,33]$ *and* $w'(N) \in [30,42]$. *The linear program of the pessimistic scenario is as follows:*

$$\begin{aligned} \text{minimize} \quad & \underline{\Gamma}_1 + \underline{\Gamma}_2 + \underline{\Gamma}_3 \\ \text{suchthat} \quad & \underline{\Gamma}_1 \geq 0, \underline{\Gamma}_2 \geq 0, \underline{\Gamma}_3 \geq 0, \\ & \underline{\Gamma}_1 + \underline{\Gamma}_2 \geq 15, \\ & \underline{\Gamma}_1 + \underline{\Gamma}_3 \geq 20, \\ & \underline{\Gamma}_2 + \underline{\Gamma}_3 \geq 25. \end{aligned}$$

The optimal solution of the pessimistic scenario is given by

$$\underline{\Gamma}_1^* = 5, \underline{\Gamma}_2^* = 10, \underline{\Gamma}_3^* = 15.$$

The linear program of the optimistic scenario is as follows:

$$\begin{aligned} \text{minimize} \quad & \overline{\Gamma}_1 + \overline{\Gamma}_2 + \overline{\Gamma}_3 \\ \text{suchthat} \quad & \overline{\Gamma}_1 \geq 0, \overline{\Gamma}_2 \geq 0, \overline{\Gamma}_3 \geq 0, \\ & \overline{\Gamma}_1 + \overline{\Gamma}_2 \geq 21, \\ & \overline{\Gamma}_1 + \overline{\Gamma}_3 \geq 30, \\ & \overline{\Gamma}_2 + \overline{\Gamma}_3 \geq 33. \end{aligned}$$

The optimal solution of the optimistic scenario is given by

$$\overline{\Gamma}_1^* = 9, \overline{\Gamma}_2^* = 12, \overline{\Gamma}_3^* = 21.$$

Then, the grey core elements of $< N, w' >$ are

$$\Gamma_1^* \in [5,9], \Gamma_2^* \in [10,12], \Gamma_3^* \in [15,21].$$

6. An application

In this section, three social network analysis companies Firm-1, Firm-2, and Firm-3, are considered. The task of these companies by providing services to the advertiser's company is to calculate the cost according to the criteria appearing on social media of advertising companies to advertise. All firms need to have the same type of information about users to analyze the frequency of online advertisement on their social media web sites. Each social network analysis company owns a storage in which it stores all the log records that may be needed. Each firm has learned number it seems online advertising in a year (**month or click**) that is automatically shown in their profile according to their interests. In the sector of social network analysis sector, holding and ordering costs change slightly from one production period to another. For example, the holding costs may be different in a holiday season. The ordering costs may be depending on the change in price of online advertising performance criteria. These criteria are CPM (Cost per Thousand Impressions), CPC (Cost per Click), CPA (Cost per Action), CPL (Cos per Lead) and CPS (Cost per Sale). Therefore, it is promosing to use grey holding and ordering parameters instead of real parameters. Table 1 illustrates the ordering and holding costs of the social network analysis firms with grey numbers.

Table 1. The ordering and holding costs of the shotgun firms with grey numbers.

	Firm-1	**Firm-2**	**Firm-3**
Demand Rates (items/per year)	$d_1^{'} \in [2500,3000]$	$d_2^{'} \in [1800,2100]$	$d_3^{'} \in [1700,1900]$
Ordering Costs (TL*/year)	$a_1^{'} \in [380,400]$	$a_2^{'} \in [380,400]$	$a_3^{'} \in [380,400]$
Holding Costs (TL*/year)	$h_1^{'} \in [0.7,0.75]$	$h_2^{'} \in [0.8,0.9]$	$h_3^{'} \in [1,1.15]$

*TL=Turkish Liras

Now, we find that the minimal cost per time unit for coalition S equals

$$c(S) = \sqrt{2a^{'}h_S^{'}\sum_{i\in S} d_i^{'}}. \tag{7}$$

Here, $a_i^{'} \in \left[\underline{a_i^{'}}, \overline{a_i^{'}}\right]$ is the grey number for ordering cost with lower bound and upper bound of $a_i^{'}$. Furthermore, $h_i^{'} \in \left[\underline{h_i^{'}}, \overline{h_i^{'}}\right]$ is the grey number for holding cost with lower bound and upper bound imposed on $h_i^{'}$. Moreover, $d_i^{'} \in \left[\underline{d_i^{'}}, \overline{d_i^{'}}\right]$ is the grey number for demand rate with lower bound and upper bound on $d_i^{'}$.

Let us clarify this application by using the algorithm which can be seen in Section 5. Let $N = \{1,2,3\}$ be the set of players and the grey coalitional values are calculated by using formula Eqn. (7). In this way, we obtain the grey coalitional values $\underline{c^{'}(N)}, \overline{c^{'}(N)}$, $\underline{c^{'}(S)}$ and $\overline{c^{'}(S)}$ by using Table 1 with the parameters as follows:

$$c(\{1\}) \in [1153.3,1341.6], c(\{2\}) \in [1046.1,1229,6], c(\{3\}) \in [1136.7,1322,1],$$
$$c(\{1,2\}) \in [1512.5,1749.3], c(\{1,3\}) \in [1494.8,1714.6], c(\{2,3\}) \in [1458.8,1697.1],$$
$$c(N) \in [1786.6,2049.4].$$

In this paper, we need the definition of the grey cost game and the grey core. Now, we introduce the definition of cooperative grey cost game.

Definition 14 *The cooperative grey cost game is an ordered pair* $< N, c^{'} >$, *where* $N = \{1,\ldots,n\}$ *is the set of players, and* $c^{'}: 2^N \to \Gamma(\mathrm{P})$ *is the characteristic function such that* $\Gamma_{\varnothing} = c^{'}(\varnothing) \in 0,0]$, *the grey payoff function value* $c^{'}(S) \in [\underline{c^{'}(S)}, \overline{c^{'}(S)}]$ *refers to the value of the grey expectation cost belonging to a coalition* $S \in 2^N$, *where* $\underline{c^{'}(S)}$ *and* $\overline{c^{'}(S)}$ *represent the maximum and minimum possible costs of the coalition* S.

In addition to the grey core $\mathrm{X}(c^{'})$ of the grey cost game $c^{'}$ is defined by

$$\mathrm{X}(c^{'}) := \left\{(\Gamma_1,\ldots,\Gamma_n) \in \Gamma(\mathrm{P})^N \mid \sum_{i\in N}\Gamma_i \approx c^{'}(N), \sum_{i\in S}\Gamma_i \div c^{'}(S), \forall S \in 2^N \setminus \{\varnothing\}\right\}.$$

In this application, we consider the grey cost game. By the definition of the grey core we know that

$$\sum_{i\in N}\Gamma_i \approx c'(N) \Leftrightarrow \sum_{i\in N}\underline{\Gamma_i} = \underline{c'(N)} \text{ and } \sum_{i\in N}\overline{\Gamma_i} = \overline{c'(N)},$$
$$\sum_{i\in S}\Gamma_i \div \overline{c'(S)} \Leftrightarrow \sum_{i\in S}\underline{\Gamma_i} \leq \underline{c'(S)} \text{ and } \sum_{i\in S}\overline{\Gamma_i} \leq \overline{c'(S)}.$$

Using the steps of the algorithm, we obtain a grey core element of the grey cost game as follows:
$\Gamma_1^* \in [1153,1341], \Gamma_2^* \in [359,408], \Gamma_3^* \in [274,300].$

7. Conclusion and outlook

In this paper, the novelty and originality of the grey solutions for cooperative grey games are introduced and studied. We also investigate the relations between these solutions. Our main result is Theorem 4.1 which contains the relation between the grey core, the grey dominance core, and the grey stable sets for cooperative grey games. We note that the characteristic function values of our approach are grey numbers defined by the lower and upper bounds of the outcoming interval numbers and result from solving general linear optimization problems. We have demonstrated that grey-valued games are a consequence of uncertain parameter values of the linear programming optimization problem, where the uncertainty of these parameter values is specified by grey numbers. What do we need when we want to solve a real-world problem by mathematical optimization? The first task which we have to do is to represent our problem by a mathematical model, i.e., a set of mathematical relationships (e.g., equalities, inequalities, logical conditions) that represent an abstraction of our real-world problem. Mathematical models for optimization usually lead to structured problems such as LP problems. In this way, we suggest a linear programming-based algorithm for finding the grey core element by employing grey numbers. Studying stable sets of a cooperative grey game can offer a further valuable extension of the theory of cooperative grey games. For further research, with the aid of numerical examples, we will demonstrate the applicability of our approach by means of a large computational study. In the meantime, we will propose LP-based algorithms for the other cooperative solutions such as nucleolus and kernel. Finally, we believe that online social networks will become very useful resources for empirical research in fields like classical game theory and grey system theory.

References

Bockarjova Z.M., Sauhats A., Vempers G. and Tereskina I., (2010). On application of the cooperative game theory to energy supply system planning. Energy Market (EEM): 7th International Conference on the European, Madrid, Spain, June 23-25, pp. 1-6.

Bondareva O.N., (1963). Certain applications of the methods of linear programming to the theory of cooperative games. *Problemly Kibernetiki*, Vol.10, pp. 119-139 (in Russian).

Chen W., Liu Z., Sun X. and Wang Y. (2010). A game-theoretic framework to identify overlapping communities in social networks. *Data Mining and Knowledge Discovery*, Vol. 21(2), pp. 224-240.

Deng J., (1982). Control problems of Grey Systems. *Systems and Control Letters*, Vol.5, pp. 288-294.

Deng J., (1985). *Grey System Fundamental Method*. Huazhong University of Science and Technology, Wuhan, China.

Ekici M., Palanci O. and Alparslan Gök S.Z. (2018). The grey Shapley value: an axiomatization, *IOP Conference Series: Materials Science and Engineering*, Vol. 300, pp. 1-8.

Fang Z. and Liu S.F., (2003). Grey matrix game model based on pure strategy. *Journal of Nanjing University of Aeronautics & Astronautics*, Vol. 35(4), pp. 441-445.

Gilles R.P., (2010). *The cooperative game theory of networks and hierarchies*. In Theory and Decision Library C, 1st edition, 44, Springer, Heidelberg.

Goel A. and Ronaghi F., (2012). A game-theoretic model of attention in social networks. *In International Workshop on Algorithms and Models for the Web-Graph*, pp. 78-92, Springer Berlin Heidelberg.

Alparslan Gök S.Z., Palanci O. and Yücesan Z. (2018). Peer Group Situations and Games with Grey Uncertainty. *Handbook of Research on Emergent Applications of Optimization Algorithms*, Chapter 11, pp. 265-278, IGI Global, USA.

Kleinberg J. and Tardos É., (2008). Balanced outcomes in social exchange networks. *In Proceedings of the fortieth annual ACM symposium on Theory of computing*, pp. 295-304, ACM.

Kose E. and Forest J.Y.L., (2015). N-person grey game, *Kybernetes*, Vol. 44(2), pp. 271-282.

Li G.P., Pan Z.S, Xiao B. and Huang L.W., (2014). Community discovery and importance analysis in social network. *Intelligent Data Analysis*, Vol. 18, pp. 495-510.

Liu S. and Lin Y., (2006). *Grey Information: Theory and Practical Applications*, Springer, London.

Liu W., Li W. and Yue K., (2007). *Intelligent Data Analysis*, Science Press, Beijing.

Moore R., (1979). *Methods and applications of interval analysis*, SIAM, Philadelphia.

Narahari Y. and Narayanam R., (2011a). Tutorial: Game Theoretic Models for Social Network Analysis.

Narayanam R. and Narahari Y., (2011b). Topologies of strategically formed social networks based on a generic value function-Allocation rule model. *Social Networks*, Vol. 33(1), pp. 56-69.

Narayanam R. and Narahari Y., (2011c). A shapley value-based approach to discover influential nodes in social networks. *IEEE Transactions on Automation Science and Engineering*, Vol. 8(1), pp. 130-147.

Palanci O., Alparslan Gök S.Z., Ergün S. and Weber G.-W., (2015). Cooperative grey games and the grey Shapley value. O*ptimization: A Journal of Mathematical Programming and Operations Research*, Vol. 64(8), pp. 1657-1668.

Roy S.K., Maity G., Weber G.W. and Alparslan Gök S.Z., (2016). Conic Scalarization Approach to Solve Multi-choice Multi-objective Transportation Problem with Interval Goal. *Annals of Operations Research (ANOR)*; DOI 10.1007/s10479-016-2283-4.

Saad W., Han Z., Debbah M., Hjorrungnes A. and Basar T., (2009). Coalitional game theory for communication networks: a tutorial, Signal Processing Magazine. *IEEE Transactions on Automation Science and Engineering*, Vol. 26, pp. 77-97.

Scott J., (2012). Social network analysis. Sage.

Shapley L.S., (1967). On balanced sets and cores. *Naval Research Logistics Quarterly*, Vol. 14, pp. 453-460.

Terlutter R. and Capella M.L., (2013). The gamification of advertising: analysis and research directions of in-game advertising, advergames, and advertising in social network games. *Journal of Advertising*, Vol. 42(2-3), pp. 95-112.

Tijs S., (2003). *Introduction to Game Theory*. Hindustan Book Agency, India.

von Neumann J. and Morgenstern O., (1944). *Theory of Games and Economic Behavior*. Princeton Univ. Press, Princeton NJ.

Zhang J.J., Wu D.S. and Olson D.L., (2005). The method of grey related analysis to multiple attribute decision making problems with interval numbers. *Mathematical and Computer Modelling*, Vol. 42(9-10), pp. 991-998.