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Analyzing bullwhip effect in supply networks under exogenous uncertainty

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Abstract
This paper explains a model for analyzing and measuring the propagation of order amplifications (i.e. bullwhip effect) for a single-product supply network topology considering exogenous uncertainty and linear and time-invariant inventory management policies for network entities. The stream of orders placed by each entity of the network is characterized assuming customer demand is ergodic. In fact, we propose an exact formula in order to measure the bullwhip effect in the addressed supply network topology considering the system in Markovian chain framework and presenting a matrix of network member relationships and relevant order sequences. The formula turns out using a mathematical method called frequency domain analysis. The major contribution of this paper is analyzing the bullwhip effect considering exogenous uncertainty in supply networks and using the Fourier transform in order to simplify the relevant calculations. We present a number of numerical examples to assess the analytical results accuracy in quantifying the bullwhip effect.

Keywords: Supply chain management; inventory; bullwhip effect; exogenous uncertainty; frequency domain analysis.

1. Introduction

Supply chain consists of a set of organizations, which cooperate in order to deliver final products and services to end customers and create value in the chain (Kouvelis, 2006). The concept of bullwhip effect first appeared by Forrester (1961). At the same time, Burbidge (2001) discussed problems with causes in detail and introduced an inventory control model regarding demand amplification. The main point on BWE studies is referred to Lee et al. (1997) research, which is an illustration of demand order amplification in supply chains. In a wider scope, supply network is characterized to include multiple sets of customers (e.g. markets) and suppliers. It should be considered that each member of a supply network may

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undertake the role of the customer of an upstream member or being the supplier of a downstream member or the both (Ouyang and Li, 2010). The bullwhip effect (BWE) refers to a phenomenon in a supply chain where the amplification in the upstream order sequences is usually greater than that of the downstream of the chain. This subject has attracted attention of both researchers and practitioners. One of the reasons which strengthen the propagation of order amplification is uncertainty. This paper focuses on exogenous uncertainties (e.g. transportation delays) which cannot be controlled by any of the network members and are independent of suppliers' inventory management policy. Extensive research has been conducted to analyze BWE. Since BWE results in huge extra operational costs for suppliers, analyzing and trying to reduce it, has turned into an important subject (Ouyang and Daganzo, 2006). We can classify all related researches on BWE into eight branches including detecting the relevant causes and recommending some solutions (Dejonckheere et al., 2003; Dejonckheere et al., 2004; Geary et al., 2004; Kim, 2008; Kouvelis, 2006; Miragliotta, 2006; Su and Wong, 2008; Wu and Katok, 2006; Zarandi et al., 2008), drawing demand models (Bayraktar et al., 2008; Chandra and Grabis, 2005, Dejonckheere, 2003; Gaalman, 2006; Gaalman and Disney, 2006; Hien et al., 2008; Hong and Ping, 2007; Luong, 2007; Luong and Nguyen, 2007; Ouyang, 2007; Zarandi, 2008; Zhang, 2004), detecting the effects of demand amplifications (Dhahri and Chabchoub, 2007; Haughton , 2009; Miragliotta, 2006), analyzing the BWE (Disney et al, 2006; Kouvelis et al., 2006; Ouyang and Li, 2010; Sucky, 2008), information sharing (Gaalman and Disney, 2006; Hsieh et al., 2007; Jaksic and Rousjan, 2007; Luong and Nguyen, 2007; Ozelkan and CakanyIldIrIm, 2009; Su and Wong, 2008; Wright and Yuan, 2008; Zhang, 2005, Zhang, 2004), channel alignment (Zhang, 2004) and operational efficiency (Bayraktar et al., 2008; Disney and Towill, 2003; Disney and Towill, 2003; Hoberg et al., 2007; Hsieh et al., 2007; Jaksic and Rousjan., 2008; Miragliotta, 2006; Ozelkan and CakanyIldIrIm, 2009; Potter, 2006; Sheu, 2005; Wright, and Yuan, 2008).

The majority of these researches use a statistical approach to derive the variance of the orders placed by the supplier under certain customer demand process.

The BWE field of study was extended to a real-world case as studying order amplification in an industrial machinery production chain (Kouvelis et al., 2006), analyzing the BWE causes (Towill et al., 2007), introducing a framework to classify BWE reduction solutions (Kouvelis et al., 2006; Towill et al., 2007; Wu and Katok 2006), studying BWE concept (Dejonckheer et a., 2003; Dejonckheer et al., 2004; Disney et al., 2006; Geary , 2004; Haughton, 2009; Towill et al., 2007; Wu and Katok 2006; Zarandi et al., 2008) and introducing order model (Gaalman, 2006; Geary , 2004, Luong, 2007, Ouyang and Daganzo, 2006; Zhang, 2004).

Kahn (1987) and Blinder (1986) studied some real-world cases, which were similar to BWE's definition with regard to macroeconomic data.

A system control framework was recently introduced to study the bullwhip effect in the frequency domain (Dejonckheere et al., 2003; Ouyang and Daganzo, 2006; Ouyang and Daganzo, 2008).

Towill et al. (1992) introduced five operational solutions to avoid BWE by means of simulation. Miragliotta (2006) proposed a model of the ordering process, later quoted by Towill et al. (1992), which was useful to isolate the BWE. Typically, BWE celebrity is certainly due to the beer distribution game, a role-playing simulation game developed at "MIT" university to illustrate the concepts of industrial dynamics by Dejonckheere et al. (2003). Disney et al. (2006) and Disney and Towill (2003) studied the BWE and introduced a model for measuring BWE magnitude. Variance Ratio is the most widely used measure to estimate BWE's value, and is defined as the ratio between the supply variance at the upstream stages and the demand variance at the downstream stages. When this ratio turns out to be higher than one, then we can conclude that there is BWE (Miragliotta, 2006).In some situations, predicting and controlling customer demand is difficult, therefore robust diagnostic tests are presented for the existence of the bullwhip effect at any stage of the chain (Ouyang and Daganzo, 2006).Since the current research follows the researches done by Ouyang and Daganzo (2006), we give a brief literature review of the mentioned research. Ouyang and Daganzo (2006) indicated how to reduce BWE by introducing advanced demand information (ADI) into the ordering schemes of supply chains. In another research, Ouyang and Daganzo (2008) analyzed the BWE in a single-echelon supply chain driven by arbitrary customer demands considering exogenous uncertainty. The supply chain was modeled as a Markovian Jump linear system with stochastic system parameters. They presented robust analytical conditions to diagnose the BWE and to bound its magnitude in supply chain under exogenous uncertainty (Ouyang, and Daganzo, 2008). Ouyang (2007) analyzed the BWE in multi-stage supply chains operated under linear and time-invariant (LTI) inventory management policies and shared supply chain information. Such information includes past order sequences and inventory records at all the stages. Recently supply chains have a network topology in which each supplier's ordering decisions influenced directly by orders from multiple neighbors, or indirectly via network-wide information sharing is an important issue. Ouyang and Li (2010) analyzed BWE in supply chain networks operated under LTI inventory management policies. They proposed a control system framework, including any supply network topologies and any stationary customer demand process.

The current research aims at analyzing BWE in a single-product supply network under exogenous uncertainty. The exogenous uncertainty instances can be transportation delays, disasters, and natural accidents whose occurrence is not under control of the supply network members. Supply network includes all members who can undertake the role of the customer of an upstream member or the supplier of a downstream one or both. Customer demand process is defined to be ergodic. We can structure the proposed method in two sections. In the first section, the supply network system is defined in the Markovian chain framework, the relationship matrix of network members is presented and then the order sequence equations are obtained. In fact, we propose a model for the supply network in the Markovian chain framework by defining stochastic dynamic parameters and exogenous uncertainty; the first section provides a basis for predicting the presence of BWE and reducing its magnitude in the supply network topology operated under LTI inventory management policies. In the second section, we give an exact formula for predicting the presence of BWE and its magnitude using frequency domain analysis (FDA) (Zarandi, 2008). This paper aims at studying the supply network system in equilibrium state and then analyzing different equations to measure the BWE by means of FDA and mainly, by Fourier transform method. The rest of the paper is structured as follows: Section 2 gives a review of supply network concept and system dynamic parameters. In Section 3, we formulate the problem and in Section 4 we present and solve a few numerical examples to indicate the performance of the proposed model. Section 5 gives conclusions and ideas for further research.

2. Supply network concept and system dynamics

Fig. 1 illustrates a general supply network with N members. The network members are classified into three sets including primitive suppliers (A_1), intermediate suppliers (A_2), and final customers (A_3), where intermediate suppliers serve final customers, and primitive suppliers serve intermediate ones. Material stream moves from primitive suppliers to intermediate ones and then to final customers in the supply network; the order stream is reverse. Primitive suppliers are assumed not to have any upstream suppliers and similarly, final customers are assumed not to have any downstream customers; intermediate suppliers can play both roles, i.e. being the customer of a primitive or an intermediate member or the supplier of an intermediate or a final customer. A directed arc $(i,k)\in N$ is applied to indicate that member $i\in A_2\cup A_3$ of the network place order at member $k\in A_1\cup A_2$. As in Ouyang and li (2010) we have assumed that A_1, A_2 and A_3 are disjoint, and the entire network consists of four disjoint subsets of arcs:

$$N_1=\{(i,k)\in N \mid i\in A_3, k\in A_1\} \tag{1}$$

$$N_2=\{(i,k)\in N \mid i\in A_3, k\in A_2\} \tag{2}$$

$$N_3=\{(i,k)\in N \mid i\in A_2, k\in A_2\} \tag{3}$$

$$N_4=\{(i,k)\in N \mid i\in A_2, k\in A_1\} \tag{4}$$

Since intermediate suppliers which are the members of A_2 can play both roles of customer and supplier, we focus on such members and identify their system dynamics. For a generic supplier $i\in A_2$, its inventory position $x_i(t)$ (including in-transit inventory) and in hand inventory $y_i(t)$. Satisfy conservations according to orders placed and received. The introduced supply network topology is supposed to be general; therefore, we can extend the results to any network structure operated under LTI inventory management policy.
LTI policy definition comes as follow: If $(i,k)\in N$, supplier *i* orders $u_{ik}(t)$ items from *k* at discrete times *t=...,-2,-1,0,1,2,...,* and receives the items after a constant lead time, $l_{ik}=0,1,2,\ldots$; assuming that upstream members always are in-stock (Dejonckheere et al.,2003; Gaalman, 2006; Kouvelis, 2006; Ouyang and Daganzo, 2006; Ouyang and Daganzo, 2008; Ouyang, 2007; Zhang, 2004). Eq. (5) and (6) define the system dynamics for the supply network.

$$x_i(t+1)=x_i(t)+\sum_{s:(i,s)\in N} u_{is}(t)-\sum_{r:(r,i)\in N} u_{ri}(t), \quad \forall i\in A_2 \tag{5}$$

$$y_i(t+1)=y_i(t)+\sum_{s:(i,s)\in N} u_{is}(t-l_{is})-\sum_{r:(r,i)\in N} u_{ri}(t), \quad \forall i\in A_2 \tag{6}$$

The order quantity placed by each member is dependent on the corresponding ordering policy; therefore, the impressive role of order policies in supply network order quantities introduces it as system dynamic. In this paper we focus on linear and time-invariant LTI policies(Wu and Katok, 2006; Zarandi et al., 2008; Zhang, 2005; Zhang, 2004).
For prospering such policies, two assumptions are considered, which are:

1- Sizes of the orders received are constant over time
2- The supplier inventories tend to equal equilibrium values that are independent of the initial conditions.

The most general LTI expression of order policy for $u_{ik}(t), (i,k)\in N_3\cup N_4$ can be stated as in Eq. (7):

$$u_{ik}(t) = \lambda_{ik} + \sum_{r \in A_2 \cup A_3} \left[A_{ik}^r(P) x_r(t) + B_{ik}^r(P) y_r(t) \right] + \sum_{r \in A_2 \cup A_3, s \in A_1 \cup A_2} \left[C_{ik}^{rs}(P) u_{rs}(t-1) \right], \quad \forall (i,k) \in N_3 \cup N_4 \tag{7}$$

Where unit shift operator P satisfies $P^m . x_i(t) = x_i(t-m), \quad \forall\, t \text{ and } \forall m = 0,1,\ldots;$ λ_{ik} is a real number, and $A_{ik}^r(P), B_{ik}^r(P)$ and $C_{ik}^{rs}(P)$ are three polynomials with real coefficients which respectively indicate how $u_{ik}(t)$ is determined based on supplier inventory history, x_r, y_r and its past orders μ_{rs} (Zhang, 2004). A general definition of $A_{ik}^r(P), B_{ik}^r(P)$ and $C_{ik}^{rs}(P)$ shows that these equations may improve any shared (or local) information, so they can represent any possible LTI ordering polices and these equations are denoted by:

$$\mathrm{A}_{\mathrm{ik}}^{\mathrm{r}}(p) = \begin{cases} -1, & if\ r = i \\ 0, & otherwise \end{cases} \quad \mathrm{B}_{\mathrm{ik}}^{\mathrm{r}}(p) = 0\ \forall r; \quad \mathrm{C}_{\mathrm{ik}}^{\mathrm{rs}}(p) = \begin{cases} \frac{l_{ik}}{m}(1 + p + .. + p^{m-1}), & if\ s = i0 \\ 0, & otherwise \end{cases}$$

System dynamics Eq. (5)-(7) make a basis in order to problem modeling.

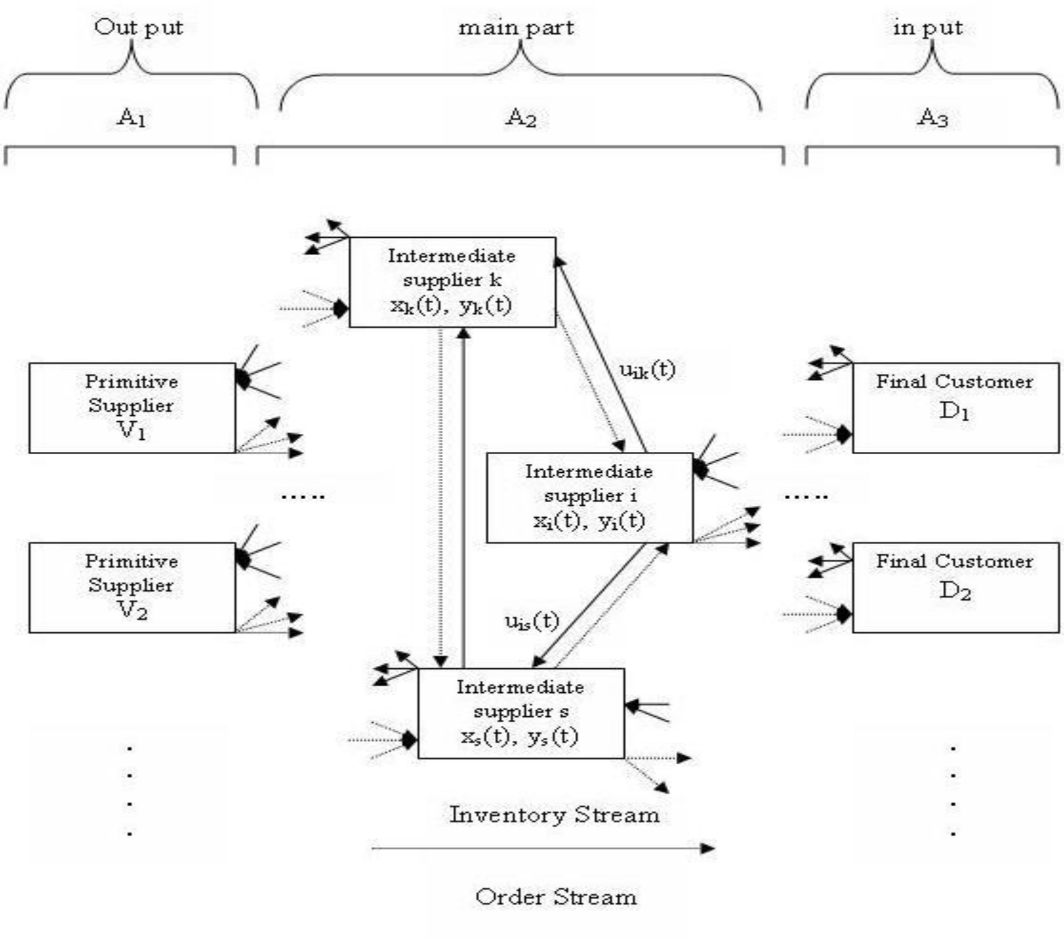

Figure 1. Supply Network Topology

We assume that LTI policies are stable; i.e. when the sizes of orders received are constant over time, the supplier inventories tend to reach equilibrium values which are independent of the initial conditions ($u_{ri}(t) = u_{ri}^{\infty}, \forall (r,i) \in N$, $u_{is}(t) = u_{is}^{\infty}, \forall (i,s) \in N$).

In the steady state, the whole orders received by supplier *i* is equal to the whole orders placed by the same supplier, i.e. $\left(\sum u_{ri}^{\infty} = \sum u_{is}^{\infty} \ , \forall i \in A_2\right)$.

Referring to the aforementioned statements, the system dynamics can be presented as in (8) and (9):

$$0 = \sum_{s:(i,s)\in N} u_{is}^{\infty} - \sum_{r:(r,i)\in N} u_{ri}^{\infty} \qquad , \forall i \in A_2 \tag{8}$$

$$u_{ik}^{\infty} = \lambda_{ik} + \sum_{r\in A_2 \cup A_3} \left[A_{ik}^{r}(P)x_r^{\infty} + B_{ik}^{r}(P)y_r^{\infty}\right] + \sum_{r\in A_2 \cup A_3, s\in A_1 \cup A_2} \left[C_{ik}^{rs}(P)u_{rs}^{\infty}\right] \qquad , \forall (i,k) \in N_3 \cup N_4 \tag{9}$$

After the definition of system dynamic, the concept of uncertainty is entered. The target of this model is formulated according to studied exogenous uncertainty which can be transportation delays, disasters, and natural accidents whose occurrence is not under the control of the supply network members. We define supply chain uncertainty as the standard deviation of the difference between the actual and expected amount. Therefore, when exogenous uncertainty is occurred, there is a deviation from steady state conditions. The major parameters can be expressed in terms of deviations from their value at the equilibrium condition that which are:

1- On-hand inventory,
2- Inventory position,
3- Order policy

That follow as in (10)-(12):

$$\overline{x_i}(t) = x_i(t) - x_i^{\infty} \tag{10}$$

$$\overline{y_i}(t) = y_i(t) - y_i^{\infty} \tag{11}$$

$$\bar{u}_{rs}(t) = u_{rs}(t) - u_{rs}^{\infty} \tag{12}$$

Correspondingly, the system dynamics under exogenous uncertainty can be represented equivalently by the following equations:

$$\bar{u}_{ik}(t) = \lambda_{ik} + \sum_{r\in A_2 \cup A_3} \left[A_{ik}^{r}(P)\bar{x}_r(t) + B_{ik}^{r}(P)\bar{y}_r(t)\right] + \sum_{r\in A_2 \cup A_3, s\in A_1 \cup A_2} \left[C_{ik}^{rs}(P)\bar{u}_{rs}(t-1)\right] \quad , \forall (i,k) \in N_3 \cup N_4 \tag{13}$$

$$\bar{x}_i(t+1) = \bar{x}_i(t) + \sum_{s:(i,s)\in N} \bar{u}_{is}(t) - \sum_{r:(r,i)\in N} \bar{u}_{ri}(t) \quad , \forall i \in A_2 \tag{14}$$

$$\bar{y}_i(t+1) = \bar{y}_i(t) + \sum_{s:(i,s)\in N} \bar{u}_{is}(t - l_{is}) - \sum_{r:(r,i)\in N} \bar{u}_{ri}(t) \quad , \forall i \in A_2 \tag{15}$$

To obtain a better comprehension of the above explanations, we summarize them in Fig. 2 as follows:

System Dynamics

$x_i(t+1) \equiv Eq.(5)$ $\quad y_i(t+1) \equiv Eq.(6)$ $\quad u_{ik}(t) \equiv Eq.(7)$

System Dynamics in Equilibrium State

$$x_i^{\infty} = x_i^{\infty} + \sum_{s:(i,s)\in N} u_{is}^{\infty} - \sum_{r:(r,i)\in N} u_{ri}^{\infty} \qquad ,\forall i \in A_2$$

$$y_i^{\infty} = y_i^{\infty} + \sum_{s:(i,s)\in N} u_{is}^{\infty} - \sum_{r:(r,i)\in N} u_{ri}^{\infty} \qquad ,\forall i \in A_2$$

$$u_{ik}^{\infty}(t) \equiv Eq.(9)$$

System Dynamics in Deviation Equilibrium State
(First entry exogenous uncertainty in model)

$\bar{x}_i(t+1) \equiv Eq.(14)$ $\quad \bar{y}_i(t+1) \equiv Eq.(15)$ $\quad \bar{u}_{ik}(t+1) \equiv Eq.(13)$

Figure 2. System dynamics

3. Formulation

3.1. Model Representation

Since the system reaches equilibrium in the steady state, we can let $\bar{x}_i(t)=\bar{y}_i(t)=\bar{u}_{rs}(t)=0,\ \forall t=-\infty,\dots,0.$ From Eq. (13)-(15), each set of downstream orders given by (16) causes a unique set of upstream order sequences given by (17).

$$\left\{\bar{u}_{rs}(t):(r,s)\in N_1\cup N_2\right\} \tag{16}$$

$$\left\{\bar{u}_{ik}(t):(i,k)\in N_3\cup N_4\right\} \tag{17}$$

Therefore, regarding to Eq. (13), (16) and (17), order sequences can be represented by the following equation:

$$\begin{aligned}\bar{u}_{ik}(t+1)-\bar{u}_{rs}(t)=\lambda_{ik}&+\sum_{r\in A_2\cup A_3}\left[A_{ik}^{r}(P)\left(\bar{x}_r(t+1)-\bar{x}_r(t)\right)+B_{ik}^{r}(P)\left(\bar{y}_r(t+1)-\bar{y}_r(t)\right)\right]\\&+\sum_{r\in A_2\cup A_3,s\in A_1\cup A_2}\left[C_{ik}^{rs}(P)\left(\bar{u}_{rs}(t)\right)\right]\qquad,\forall(i,k)\in N_3\cup N_4\end{aligned} \tag{18}$$

Using (14) and (15) to eliminate $\bar{x}_r(t+1)-\bar{x}_r(t)$ and $\bar{y}_r(t+1)-\bar{y}_r(t)$, $\bar{u}_{ik}(t+1)$ can be obtained from the following equation.

$$\bar{u}_{ik}(t+1)=\Psi(P).\bar{u}_{ik}(t)+\Phi(P).\bar{u}_{rs}(t), \tag{19}$$

Where, $\Psi(P)$ and $\Phi(P)$ are polynomials with finite degrees, as represented in Eq. (20) and are referred to $\sum\left[A_{ik}^{r}(P)+B_{ik}^{r}(P)\right]$ and $\sum\left[C_{ik}^{rs}(P)\right]$, respectively.

$$K=Max\{\deg\Psi(P),\deg\Phi(P)\}\leq\infty \tag{20}$$

Here, a summary of the aforementioned statements and equations is presented in Fig. 3:

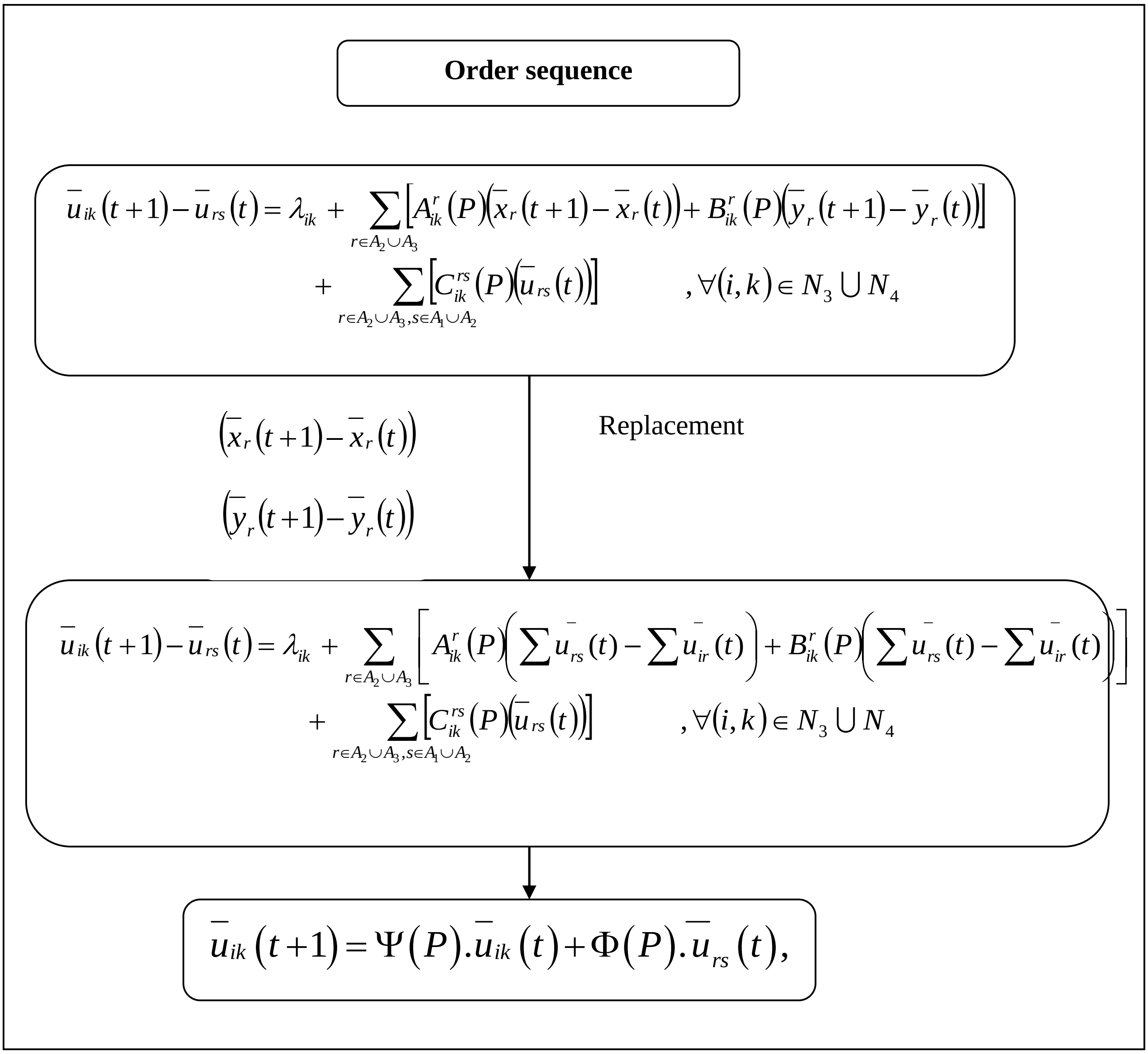

Figure 3. Order sequences

Since we should represent the order sequences to define supply network members' order interactions, we need to represent the matrix form of the orders. Eq. (21) gives the matrix form representation of Eq. (19)

$$\mathrm{u}_{ik}(t+1)=\mathrm{r}_{ik}.\mathrm{u}_{ik}(t)+\mathrm{s}_{rs}.\mathrm{u}_{rs}(t), \tag{21}$$

Where r_{ik} and s_{rs} are the matrix form representation of $\Psi(P)$ and $\Phi(P)$, and $\mathrm{u}_{ik}(t+1)$, $\mathrm{u}_{ik}(t)$ and $\mathrm{u}_{rs}(t)$ are the matrix representations of $\bar{u}_{ik}(t+1)$, $\bar{u}_{ik}(t)$ and $\bar{u}_{rs}(t)$, respectively. Eq. (21) presents the effect of deviation of orders placed by supplier *i* from *k* $\{(i,k)\in N_3\cup N_4\}$ and by supplier *r* from *s* $\{(r,s)\in N_1\cup N_2\}$ at time *t* on the deviation of orders placed by supplier *i* from *k* $\{(i,k)\in N_3\cup N_4\}$ at time *t+1*; therefore, Eq. (21) is just represented at state *t*. To identify system dynamics, we represent Eq. (21) in terms of multiple states $\{i.e., t, t-1, t-2, \ldots\}$;

therefore, we define $(K+1)\times 1$ column vector for two unique members $\{i,k\}$ in the supply network given by (22)-(24).

$$\mathrm{u}_{IK}(t+1)=\left[\bar{u}ik(t+1),\bar{u}ik(t),\ldots,\bar{u}ik(t-K-1)\right]^{T} \quad \text{for two unique members} \tag{22}$$

$$\mathrm{u}_{IK}(t)=\left[\bar{u}ik(t),\bar{u}ik(t-1),\ldots,\bar{u}ik(t-K)\right]^{T} \quad \text{for two unique members} \tag{23}$$

$$\mathrm{u}_{RS}(t)=\left[\bar{u}rs(t),\bar{u}rs(t-1),\ldots,\bar{u}rs(t-K)\right]^{T} \quad \text{for two unique members} \tag{24}$$

The system dynamic (21) can be written in terms of multiple states:

$$\mathrm{u}_{IK}(t+1)=\mathrm{R}_{IK}.\mathrm{u}_{IK}(t)+\mathrm{S}_{RS}.\mathrm{u}_{RS}(t), \tag{25}$$

Where R_{IK} and S_{RS} are matrices which include the effect of multiple states introduced in (21). To augment the state, the paper represents system dynamics (25) for entire supply network not just for two unique members of supply network.IT obtains:

$$\mathrm{U}_{IK}(t)=\left[\mathrm{u}_{IK}(t)^{T},\ldots,\mathrm{u}_{FG}(t)^{T}\right]^{T}. \qquad ,\forall(I,K)\in N_3\cup N_4 \tag{26}$$

$$\mathrm{U}_{RS}(t)=\left[\mathrm{u}_{RS}(t)^{T},\ldots,\mathrm{u}_{MN}(t)^{T}\right]^{T} \qquad ,\forall(R,S)\in N_1\cup N_2 \tag{27}$$

$$\mathrm{U}_{IK}(t+1)=R_{IK}.\mathrm{U}_{IK}(t)+S_{RS}.\mathrm{U}_{RS}(t) \tag{28}$$

Where, R_{IK} and S_{RS} are square matrices $\{(K+1)\times(K+1)\}$ including a number of sub-matrices $\{i.e.,\mathrm{R}_{IK},\mathrm{S}_{RS}\}$ as their elements.

A summary of the explanations is indicated by Fig. 4:

Matrix presentation of order sequences

$$\mathrm{u}_{ik}(t+1)=\mathrm{r}_{ik}.\mathrm{u}_{ik}(t)+\mathrm{s}_{rs}.\mathrm{u}_{rs}(t),$$

$$\mathrm{u}_{IK}(t+1)=\left[\bar{u}_{ik}(t+1),\bar{u}_{ik}(t),\ldots,\bar{u}_{ik}(t-K-1)\right]^T$$
$$\mathrm{u}_{RS}(t)=\left[\bar{u}_{rs}(t),\bar{u}_{rs}(t-1),\ldots,\bar{u}_{rs}(t-K)\right]^T$$
$$\mathrm{u}_{IK}(t)=\left[\bar{u}_{ik}(t),\bar{u}_{ik}(t-1),\ldots,\bar{u}_{ik}(t-K)\right]^T$$

$$R_{IK}=\begin{array}{c c} & \begin{array}{cccc} t & t-1 & \ldots\ldots & t-n \end{array} \\ \begin{array}{c} \bar{u}_{ik}(t) \\ \bar{u}_{ik}(t-1) \\ . \\ . \\ \bar{u}_{ik}(t-k) \end{array} & \begin{bmatrix} \alpha_{It} & \alpha_{\mathrm{I,t-1}} & \ldots\ldots & \alpha_{\mathrm{I,t-n}} \\ \alpha_{\mathrm{I-1,t}} & \alpha_{\mathrm{I-1,t-1}} & \ldots\ldots & \alpha_{\mathrm{I-1,t-n}} \\ . & . & \ldots\ldots & . \\ . & . & \ldots\ldots & . \\ \alpha_{\mathrm{I-n,t}} & \alpha_{\mathrm{I-n,t-1}} & \ldots\ldots & \alpha_{\mathrm{I-n,t-n}} \end{bmatrix}_{(k+1)*(K+1)} \end{array}$$

$$S_{rs}=\begin{array}{c c} & \begin{array}{cccc} t & t-1 & \ldots\ldots & t-n \end{array} \\ \begin{array}{c} \bar{u}_{rs}(t) \\ \bar{u}_{rs}(t-1) \\ . \\ . \\ \bar{u}_{rs}(t-k) \end{array} & \begin{bmatrix} \beta_{It} & \beta_{\mathrm{I,t-1}} & \ldots\ldots & \beta_{\mathrm{I,t-n}} \\ \beta_{\mathrm{I-1,t}} & \beta_{\mathrm{I-1,t-1}} & \ldots\ldots & \beta_{\mathrm{I-1,t-n}} \\ . & . & \ldots\ldots & . \\ . & . & \ldots\ldots & . \\ \beta_{\mathrm{I-n,t}} & \beta_{\mathrm{I-n,t-1}} & \ldots\ldots & \beta_{\mathrm{I-n,t-n}} \end{bmatrix}_{(k+1)*(K+1)} \end{array}$$

$$\mathrm{u}_{IK}(t+1)=\mathrm{R}_{IK}.\mathrm{u}_{IK}(t)+\mathrm{S}_{RS}.\mathrm{u}_{RS}(t),$$

$$\mathrm{U}_{IK}(t+1)=R_{IK}.\mathrm{U}_{IK}(t)+S_{RS}.\mathrm{U}_{RS}(t)$$

Figure 4. Matrix presentation of order sequences

The state space of the proposed model composed by the effect coefficient matrices {i.e., R_{IK} and S_{RS} } for entire supply network, is of multiple dimensions and can be represented in matrix pairs given by Eq. (29).

$$M=\{(\mathrm{R}_{IK},\mathrm{S}_{RS}),(\mathrm{R}_{I-1,K-1},\mathrm{S}_{R-1,S-1}),\ldots,(\mathrm{R}_{I-M,K-M},\mathrm{S}_{R-M,S-M})\} \qquad n(M)=M \tag{29}$$

It should be noted that the number of state space members is equivalent to the number of matrix pairs in the supply network, which affect order relationships. System dynamics (28), can be presented by the following equation with exogenous uncertainty:

$$\mathrm{U}_{IK}(t+1)=R_{IK(t)}.\mathrm{U}_{IK}(t)+S_{RS(t)}.\mathrm{U}_{RS}(t) \tag{30}$$

$R_{IK(t)}$ and $S_{RS(t)}$ include the matrices of R_{IK} and S_{RS} at multiple stages of the state space. In other hands, the transaction probability matrix of supply network is defined as follows:

$$\rho=[P_{mn}]_{M\times N} \quad ; P_{mn}=\Pr\{(t+1)=n|t=m\} \qquad ,\forall\, m,n\in M \tag{31}$$

Where its degree can be represented as:

$$A=\{(i,k)|(i,k)\in N_3\cup N_4\} \tag{32}$$

$$B=\{(r,s)|((r,s)\in N_1\cup N_2\} \tag{33}$$

Transaction probability matrix degree $=Max\{n(A),\, n(B)\}$ (34)

The stochastic order relationship equation (30) is useful to model any supply network topologies under exogenous uncertainty, where $R_{IK(t)}$ and $S_{RS(t)}$ can be stated as in (35) and (36):

$$R_{IK(t)}=\begin{array}{c|c} & \begin{array}{cccc} K & K-1 & \ldots\ldots & K-n \end{array} \\ \begin{array}{c} I \\ I-1 \\ . \\ . \\ I-n \end{array} & \begin{bmatrix} \mathrm{R}_{\mathrm{IK}} & \mathrm{R}_{\mathrm{I,K-1}} & \ldots\ldots & \mathrm{R}_{\mathrm{I,K-n}} \\ \mathrm{R}_{\mathrm{I-1,K}} & \mathrm{R}_{\mathrm{I-1,K-1}} & \ldots\ldots & \mathrm{R}_{\mathrm{I-1,K-n}} \\ . & . & \ldots\ldots & . \\ . & . & \ldots\ldots & . \\ \mathrm{R}_{\mathrm{I-n,K}} & \mathrm{R}_{\mathrm{I-n,K-1}} & \ldots\ldots & \mathrm{R}_{\mathrm{I-n,K-n}} \end{bmatrix}_{n\times n} \end{array} \tag{35}$$

$$S_{RS(t)}=\begin{array}{c|c} & \begin{array}{cccc} \mathrm{S} & \mathrm{S}-1 & \ldots\ldots & \mathrm{S}-n \end{array} \\ \begin{array}{c} \mathrm{R} \\ \mathrm{R}-1 \\ . \\ . \\ \mathrm{R}-n \end{array} & \begin{bmatrix} \mathrm{S}_{\mathrm{RS}} & \mathrm{S}_{\mathrm{R,S-1}} & \ldots\ldots & \mathrm{S}_{\mathrm{R,S-n}} \\ \mathrm{S}_{\mathrm{R-1,S}} & \mathrm{S}_{\mathrm{R-1,S-1}} & \ldots\ldots & \mathrm{S}_{\mathrm{R-1,S-n}} \\ . & . & \ldots\ldots & . \\ . & . & \ldots\ldots & . \\ \mathrm{S}_{\mathrm{R-n,S}} & \mathrm{S}_{\mathrm{R-n,S-1}} & \ldots\ldots & \mathrm{S}_{\mathrm{R-n,S-n}} \end{bmatrix}_{n\times n} \end{array} \tag{36}$$

Where, the relation between each pair of row and column is represented. For example, in matrix (36) the element in the first row and second column ($\mathrm{S}_{\mathrm{R,S-1}}$), presents the effect of deviation of orders placed by supplier R from S-1 on the mentioned supplier (i.e. supplier I) order deviation at the relevant state. Respectively, any elements of matrices (35) and (36) can be expressed as in (37)-(38):

$$\mathrm{R}_{IK}=\begin{array}{c|c} & \begin{array}{cccc} t & t-1 & \ldots\ldots & t-K \end{array} \\ \begin{array}{c} \bar{u}_{ik}(t) \\ \bar{u}_{ik}(t-1) \\ . \\ . \\ \bar{u}_{ik}(t-K) \end{array} & \begin{bmatrix} \alpha_{00} & \alpha_{01} & \ldots\ldots & \alpha_{0\mathrm{K}} \\ \alpha_{10} & \alpha_{11} & \ldots\ldots & \alpha_{1\mathrm{K}} \\ . & . & \ldots\ldots & . \\ . & . & \ldots\ldots & . \\ \alpha_{\mathrm{K}0} & \alpha_{\mathrm{K}1} & \ldots\ldots & \alpha_{\mathrm{KK}} \end{bmatrix}_{(K+1)\times(K+1)} \end{array} \tag{37}$$

$$\mathrm{S}_{RS} = \begin{array}{c c} & \begin{array}{cccc} t & t-1 & \ldots\ldots & t-K \end{array} \\ \begin{array}{c} \bar{u}_{rs}(t) \\ \bar{u}_{rs}(t-1) \\ \vdots \\ \\ \bar{u}_{rs}(t-K) \end{array} & \begin{bmatrix} \beta_{00} & \beta_{01} & \ldots\ldots & \beta_{0K} \\ \beta_{10} & \beta_{11} & \ldots\ldots & \beta_{1K} \\ \cdot & \cdot & \ldots\ldots & \cdot \\ \cdot & \cdot & \ldots\ldots & \cdot \\ \beta_{K0} & \beta_{K1} & \ldots\ldots & \beta_{KK} \end{bmatrix}_{(K+1)\times(K+1)} \end{array} \tag{38}$$

The matrices (37) and (38) can play role as any elements of matrices (35) and (36) at multiple defined states (focus on one supplier to analyze the BWE magnitude in that stage). The given procedure is represented by Fig. 5:

Definition of stochastic order relationship equation (Exogenous Uncertainty)

$$M = \{(\mathrm{R}_{IK}, \mathrm{S}_{RS}), (\mathrm{R}_{I-1,K-1}, \mathrm{S}_{R-1,S-1}), \ldots, (\mathrm{R}_{I-M,K-M}, \mathrm{S}_{R-M,S-M})\} \qquad n(M) = M$$

$$R_{IK(t)} = \begin{array}{c c} & \begin{array}{cccc} K & K-1 & \ldots\ldots & K-n \end{array} \\ \begin{array}{c} I \\ I-1 \\ \vdots \\ \\ I-n \end{array} & \begin{bmatrix} \mathrm{R}_{\mathrm{IK}} & \mathrm{R}_{\mathrm{I,K-1}} & \ldots\ldots & \mathrm{R}_{\mathrm{I,K-n}} \\ \mathrm{R}_{\mathrm{I-1,K}} & \mathrm{R}_{\mathrm{I-1,K-1}} & \ldots\ldots & \mathrm{R}_{\mathrm{I-1,K-n}} \\ \cdot & \cdot & \ldots\ldots & \cdot \\ \cdot & \cdot & \ldots\ldots & \cdot \\ \mathrm{R}_{\mathrm{I-n,K}} & \mathrm{R}_{\mathrm{I-n,K-1}} & \ldots\ldots & \mathrm{R}_{\mathrm{I-n,K-n}} \end{bmatrix}_{n\times n} \end{array}$$

$$S_{RS(t)} = \begin{array}{c c} & \begin{array}{cccc} \mathrm{S} & \mathrm{S}-1 & \ldots\ldots & \mathrm{S}-n \end{array} \\ \begin{array}{c} \mathrm{R} \\ \mathrm{R}-1 \\ \vdots \\ \\ \mathrm{R}-n \end{array} & \begin{bmatrix} \mathrm{S}_{\mathrm{RS}} & \mathrm{S}_{\mathrm{R,S-1}} & \ldots\ldots & \mathrm{S}_{\mathrm{R,S-n}} \\ \mathrm{S}_{\mathrm{R-1,S}} & \mathrm{S}_{\mathrm{R-1,S-1}} & \ldots\ldots & \mathrm{S}_{\mathrm{R-1,S-n}} \\ \cdot & \cdot & \ldots\ldots & \cdot \\ \cdot & \cdot & \ldots\ldots & \cdot \\ \mathrm{S}_{\mathrm{R-n,S}} & \mathrm{S}_{\mathrm{R-n,S-1}} & \ldots\ldots & \mathrm{S}_{\mathrm{R-n,S-n}} \end{bmatrix}_{n\times n} \end{array}$$

$$\mathrm{U}_{IK}(t+1) = R_{IK(t)}.\mathrm{U}_{IK}(t) + S_{RS(t)}.\mathrm{U}_{RS}(t)$$

Figure 5. Exogenous uncertainty

3.2. Bullwhip effect Metric

An easily understood metric to measure the BWE is the ratio of root mean square errors (RMSE) of a supplier order sequence to RMSE of a customer demand sequence in a simple supplier-customer relationship. Since the RMSE factor has been applied by various researches to measure the BWE (Dejonckheere, 2003; Gaalman, 2006; Gear et al., 2004; Kouvelis et al., 2006; Luong, 2007; Miragliotta, 2006; Ouyang, and Daganzo, 2006; Ouyang and Daganzo, 2008; Ouyang and li, 2010; Ouyang, 2007), We apply the RMSE factor (W) to measure the BWE magnitude.

$$RMSE = \left[\frac{\sum (x-\mu)^2}{n} \right]^{1/2} \tag{39}$$

Having calculated order deviation of down streams and up streams through (40) and (41) as is stated in the following, we can compute the BWE as in Eq. (42) (Miragliotta, 2006; Ouyang, and Daganzo, 2006; Ouyang and Daganzo, 2008; Ouyang and li, 2010; Ouyang, 2007):

$$\bar{u}_D(t) = \sum_{\forall (r,s) \in N_1 \cup N_2} \bar{u}_{rs}(t) \;\; \forall t, \tag{40}$$

$$\bar{u}_V(t) = \sum_{\forall (r,s) \in N_1 \cup N_4} \bar{u}_{rs}(t) \;\; \forall t, \tag{41}$$

$$W = \left[\frac{\sum_{t=0}^{\infty} \bar{u}_V^2(t)}{\sum_{t=0}^{\infty} \bar{u}_D^2(t)} \right]^{1/2} \qquad \forall \left\{ \bar{u}_D(t) \neq 0 \right\} \tag{42}$$

The condition $W \leq 1$ guarantees that the RMSE is not amplified (i.e. BWE dose not arise) under any customer ordering scenario (Ouyang and li, 2010). Meanwhile, we utilize frequency domain analysis, specially, Fourier transform, in order to measure BWE on the basis of RMSE.

The Frequency domain analysis has been used to study the properties of time series for a long time (Ouyang and li, 2010; Ouyang, 2007). The standard techniques (e.g. transformation) significantly facilitate quantification of a linear system's response to input signals (Ouyang and li, 2010). The supply network can be regarded as a multiple-input, multiple-output system, with all final customer demands as the inputs, and all primitive supplier orders as the outputs. Regarding Ouyang and li (2010) the BWE magnitude (*W*) can be expressed equivalently in the frequency domain (Wu and Katok (2006); ; Zarandi et al. (2008); Zhang , 2005; Zhang , 2005)any set of customer orders [$\{\bar{u}_{rs}(t)\}, \forall (r,s) \in N_1 \cup N_2$] can be presented by a discrete Fourier transform [$A_{rs}(w) e^{iwt} \quad w \in [-\pi, \pi) \quad (i = \sqrt{-1})$]. Since system dynamics are LTI, it is clear that for any harmonic component of any customer demand (as system input), $\left[A_{rs}(w) e^{iwt}\right]$, the resulting orders placed by each supplier of the LTI network (as system output) are also harmonic (regarding Fourier transform definition) with the same frequency $\left[A_{ik}^{rs}(w) e^{iwt}\right]$. Fourier transform presentation of all received orders by A_1 can be written as in (43):

$$\left\{A_{ik}^{rs}(w) e^{iwt} \middle| \forall (i,k) \in N_4, \forall (r,s) \in N_1 \cup N_2, w \in [-\pi, \pi)\right\} \cup \left\{A_{rs}(w) e^{iwt} \middle| \forall (r,s) \in N_1, w \in [-\pi, \pi)\right\}, \tag{43}$$

The first set represents received orders by A_1 from A_2 placed by A_3 and the second set represents received orders by A_1 placed directly by A_3; therefore, it is clear that the union of the two mentioned sets represents all received orders by A_1.

Fourier transform representation of all placed orders by A_3 is given by (44):

$$A_{rs}(w) e^{iwt} \;\; \forall (r,s) \in N_1 \cup N_2, w \in [-\pi, \pi) \tag{44}$$

Regarding system matrix representation and order variances, the ratio of output equations and input equations is required in order to define transfer function matrix which is given by (45).

$$T_{ik}^{rs}(w) = \frac{A_{ik}^{rs}(w)}{A_{rs}(w)} \quad \forall (i,k) \in N_1 \cup N_4, \forall (r,s) \in N_1 \cup N_2 \tag{45}$$

The transfer function matrix is introduced focusing on A_2 as multiple-input, multiple-output system. As Ouyang et al. methodology, we use z-transform to obtain transfer function matrix in its own way (Ouyang and Daganzo, 2006; Ouyang and Daganzo, 2008; Ouyang and Li, 2010). Denote the z-transforms of different equations ($X_i(z)=z\{\bar{x}_i(t)\}, \forall i \in A_2$, $Y_i(z)=z\{\bar{y}_i(t)\}, \forall i \in A_2$ and $U_{ik}(z)=z\{\bar{u}_{ik}(t)\}, \forall (i,k) \in N$) for all polynomials introduced in (7):

$$z\{A_{ik}^{r}(P)\bar{x}_r(t)\}=A_{ik}^{r}(z^{-1})X_r(z) \tag{46}$$

$$z\{B_{ik}^{r}(P)\bar{y}_r(t)\}=B_{ik}^{r}(z^{-1})Y_r(z) \tag{47}$$

$$z\{C_{ik}^{rs}(P)\bar{u}_{rs}(t-1)\}=C_{ik}^{rs}(z^{-1})z^{-1}U_{rs}(z) \tag{48}$$

if we apply z-transform to both sides of system dynamics equations, we can obtain:

$$U_{ik}(z)=\sum\left[A_{ik}^{r}(z^{-1})X_r(z)+B_{ik}^{r}(z^{-1})Y_r(z)\right]+z^{-1}\sum\left[C_{ik}^{rs}(z^{-1})U_{rs}(z)\right] \quad ,\forall (r,s)\in N\ ,\forall r\in A_2 \tag{49}$$

$$(z-1)X_i(z)=\sum U_{is}(z)-\sum U_{ri}(z)+z\bar{x}_i(0) \tag{50}$$

$$(z-1)Y_i(z)=\sum z^{-1}U_{is}(z)-\sum U_{ri}(z)+z\bar{y}_i(0) \tag{51}$$

We assume the system state after the equilibrium state at t=0, when we have $\bar{x}_i(0)=\bar{y}_i(0)=\bar{u}_{rs}(0)=0, \forall i, \forall (r,s)$. Since we aim at analyzing BWE, this would satisfy the stochastic order relationship equation (30) by z-transform. Thus, in order to apply z-transform, the relevant steps can be stated as follows:

a) The elements of (19) can be represented using z-transform as follows:

$$z\{\bar{u}_{ik}(t+1)\}=(z-1)U_{ik}(z) \tag{52}$$

$$z\{\Psi(P).\bar{u}_{ik}(t)\}=\Psi(z^{-1}).U_{ik}(z) \tag{53}$$

$$z\{\Phi(P).\bar{u}_{rs}(t)\}=\Phi(z^{-1}).U_{rs}(z) \tag{54}$$

Equation (19) comes out as in (55):

$$(z-1)U_{ik}(z)=\Psi(z^{-1}).U_{ik}(z)+\Phi(z^{-1}).U_{rs}(z) \tag{55}$$

b) Noting the definition of matrices (22)-(24), Eq. (25) can be expressed as in (56) using z-transform:

$$(z-1)U_{IK}(z)=\mathrm{R}_{\mathrm{IK}}.U_{IK}(z)+\mathrm{S}_{RS}.U_{RS}(z) \tag{56}$$

c) The z-transform representation of Eq. (30) is as in (57):

$$(z-1)U_{IK}(z)=R_{\mathrm{IK}(z)}.U_{IK}(z)+\mathrm{S}_{RS(z)}.U_{RS}(z) \tag{57}$$

$$(z-1)M\,U_{IK}(z)=R_{\mathrm{IK}(z)}.U_{IK}(z)+\mathrm{S}_{RS(z)}.U_{RS}(z) \tag{58}$$

$$\left[(z-1)M-R_{IK(z)}\right]U_{IK}(z)=S_{RS(z)}.U_{RS}(z) \tag{59}$$

Where, M is an identity matrix and $\left|(z-1)M-R_{IK(z)}\right|$ is invertible; we can obtain $U_{IK}(z)$ from (59) as in (60):

$$U_{IK}(z)=\left(\frac{S_{RS(z)}}{(z-1)M-R_{IK(z)}}\right).U_{RS}(z) \tag{60}$$

Where $\left(\frac{S_{RS(z)}}{(z-1)M - R_{IK(z)}}\right)$ is replaced by *T (z)* as in (61):

$$U_{IK}(z) = T(z).U_{RS}(z) \tag{61}$$

Each element in the transfer function matrix is applied to indicate that how customer demands transform to supplier orders. Referring to Parseval's Theorem[1] [2], the order flow variance of the two specified members can be presented by (62):

$$\frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \left| U_{IK}\left(e^{iw}\right) \right|^2 dw \tag{62}$$

Where e^{iw} can be replaced with "z" (i.e. approximating z-transform by Fourier transform). Eq. (62) can be applied as in (63) in order to calculate the ratio of supplier order sequences to customer demand sequences (*W*) for each pair of the supply network.

$$W = \left[\frac{\int_{-\pi}^{\pi} \left| U_{IK}\left(e^{iw}\right) \right|^2 dw}{\int_{-\pi}^{\pi} \left| U_{RS}\left(e^{iw}\right) \right|^2 dw} \right]^{1/2} \tag{63}$$

W is the ration of order received by up-streams and order placed by down-streams, represented by Fourier transform. Applying Eq. (63) helps to analyze the BWE magnitude and the BWE can be ignored if the *W* turns out to be less than 1 as is declared by previous researchers (Dejonckheere et al., 2003; Gaalman , 2006; Geary et al., 2004; Kouvelis et al., 2006).
In order to present better sense of paper procedure, this section is summarized by Fig. 6:

[1] $\frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathcal{F}(\omega)|^2 \, d\omega$

Extracting Transfer function matrix to reach robust bullwhip effect analyzing formula

$$(x-1)U_{IK}(x) = \mathrm{R}_{\mathrm{IK}}.U_{IK}(x) + \mathrm{S}_{RS}.U_{RS}(x)$$

$$\left[(z-1)M - R_{IK(z)}\right]U_{IK}(z) = S_{RS(z)}.U_{RS}(z)$$

$$U_{IK}(z) = \left(\frac{S_{RS(z)}}{(z-1)M - R_{IK(z)}}\right).U_{RS}(z)$$

$$T(z) = \left(\frac{S_{RS(z)}}{(z-1)M - R_{IK(z)}}\right)$$

$$U_{IK}(z) = T(z).U_{RS}(z)$$

$$W = \left[\frac{\int_{-\pi}^{\pi}\left|\mathrm{U}_{IK}\left(e^{iw}\right)\right|^{2} dw}{\int_{-\pi}^{\pi}\left|\mathrm{U}_{RS}\left(e^{iw}\right)\right|^{2} dw}\right]^{1/2}$$

Figure 6. Measuring BWE

Numerical Examples

In this section, we consider a supply network consisting of six members as given in Fig.7; two final customers, three intermediate suppliers and a primitive supplier. As declared in the previous sections, A_1, A_2 and A_3 are disjoint; therefore, the network includes four disjoint subsets of arcs as in (64)-(67):

$$N_1 = \{\ \} \tag{64}$$

$$N_2 = \{(1,3),(2,4)\} \tag{65}$$

$$N_3 = \{(3,5),(4,5)\} \tag{66}$$

$$N_4 = \{(5,6)\} \tag{67}$$

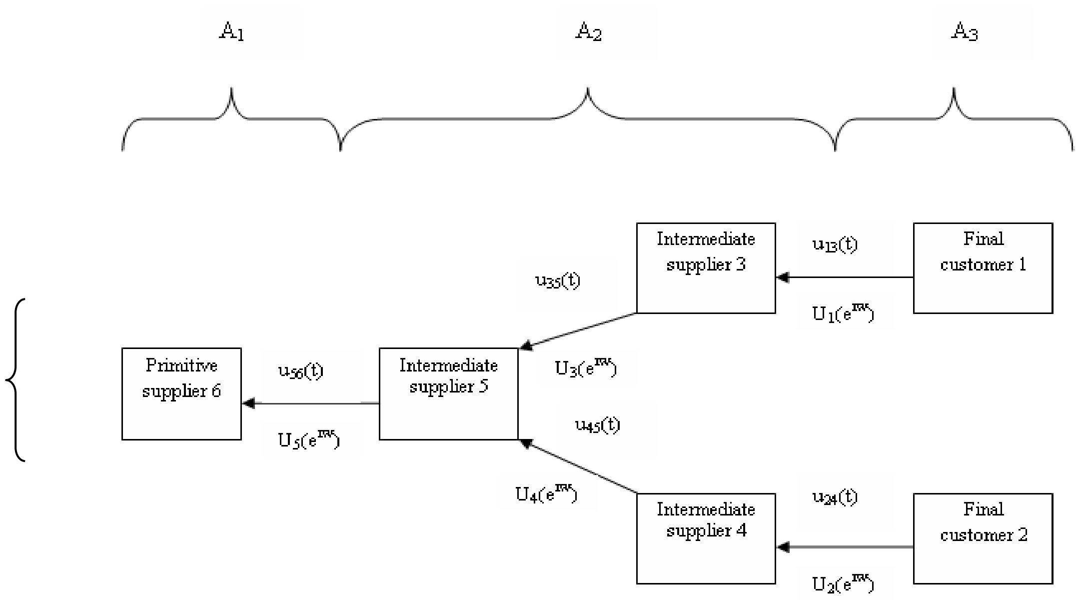

Figure 7. Given supply network

Lead-time is assumed constant. It is assumed that all network members use (S, s) ordering policy with no information sharing and demand is predicted using moving-average of orders received within two recent periods. In this case, all the analysis and results can be obtained from focusing on an intermediate supplier; we consider supplier 3 for this purpose (the given analysis can be extended focusing on any member of the supply network). The order sequence equation can be written as in (68):

$$\bar{u}35(t+1) = 2\bar{u}35(t) + 1.5\bar{u}35(t-1) + 3.1\bar{u}13(t) + 3\bar{u}13(t-1) \tag{68}$$

where, it is simply obtained from the following system dynamic metrics:

$$x_3(t+1) = x_3(t) + u_{35}(t) - u_{13}(t) \tag{69}$$

$$u_{35}(t) = \lambda_{35} - x_3(t) + \left(\frac{2+2P}{2}\right).u_{13}(t-1) \tag{70}$$

$$u_{35}(t)=\lambda_{35}-x_3(t)+u_{13}(t-1)+P.u_{13}(t) \tag{71}$$

$$u_{35}(t)=\lambda_{35}-x_3(t)+u_{13}(t-1)+u_{13}(t-2) \tag{72}$$

Given the assumed network topology, relations matrices given by {(26) and (27)} are represented as following:

$$U_{IK}(t)=\left[u_{35}(t)^T, u_{45}(t)^T, u_{56}(t)^T\right]^T_{1\times 3} \quad (I,K)\in N_3\cup N_4 \tag{73}$$

$$U_{RS}(t)=\left[u_{13}(t)^T, u_{24}(t)^T\right]^T_{1\times 2} \qquad (R,S)\in N_2 \tag{74}$$

The matrices (69) and (70) consist of a number of sub-matrices which present the amount of order deviation from equilibrium state (i.e. $u_{35}(t)=\left[\bar{u}_{35}(t),\bar{u}_{35}(t-1),\ldots\right]^T$). We need to define $R_{IK(t)}$ and $S_{RS(t)}$ in order to interfere the effect of exogenous uncertainty.

$$R_{IK(z)}=\begin{array}{c c} & \begin{array}{ccc} 5 & 6 & \# \end{array} \\ \begin{array}{c} 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} \mathrm{R}_{35} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3} \end{array} \tag{75}$$

$$S_{RS(z)}=\begin{array}{c c} & \begin{array}{ccc} 3 & 4 & \# \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \# \end{array} & \begin{bmatrix} \mathrm{S}_{13} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3} \end{array} \tag{76}$$

The symbol (#) is used to control the number of matrices rows and columns to prevent any problems in calculations. Matrices R_{35} and S_{13} can be presented as follow:

$$\mathrm{R}_{35}=\begin{bmatrix} 2 & 0 \\ 0 & 1.5 \end{bmatrix} \tag{77}$$

$$\mathrm{S}_{13}=\begin{bmatrix} 3.1 & 0 \\ 0 & 3 \end{bmatrix} \tag{78}$$

The frequency domain representation of the entire network orders can be represented by (79)-(80).

$$U_{RS}(z)=\begin{bmatrix} U_1(z) \\ U_2(z) \\ DP \end{bmatrix} \tag{79}$$

$$U_{IK}(z)=\begin{bmatrix} U_3(z) \\ U_4(z) \\ U_5(z) \end{bmatrix} \tag{80}$$

The values of the elements of Matrix (79) are considered as follows (Dejonckheere et al., 2004; Gaalman , 2006; Zhang , 2005):

$$U_{RS}(z)=\left[[2 \quad 5]^T,[1 \quad 2.5]^T,[0 \quad 0]^T\right]^T \tag{81}$$

Using Eq. (60), the matrix $U_{IK}(z)$ and the probability function matrix {*T(z)*} can be obtained as (82)-(83):

$$T(z)=\begin{bmatrix} \begin{bmatrix} -1.03 & 0 \\ 0 & -1.20 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \tag{82}$$

$$U_{IK}(z)=\left[[-2.07 \quad -6]^T,[0 \quad 0]^T,[0 \quad 0]^T\right]^T \tag{83}$$

Since z is equivalent to e^{iw}, focusing on intermediate supplier 3, the BWE magnitude is calculated as:

$$W=\left[\frac{\int_{-\pi}^{\pi}\left|U_{IK}\left(e^{iw}\right)\right|^2 dw}{\int_{-\pi}^{\pi}\left|U_{RS}\left(e^{iw}\right)\right|^2 dw}\right]^{1/2}=1.03 \tag{84}$$

where, the amount of W[1], determines the presence of BWE in supply network (W gets greater than 1) and its magnitude is equivalent to 1.03. To have a review on the numerical analysis, you can refer to Fig.8 and Fig.9.

[1] The relevant analysis are calculated by MATLAB software

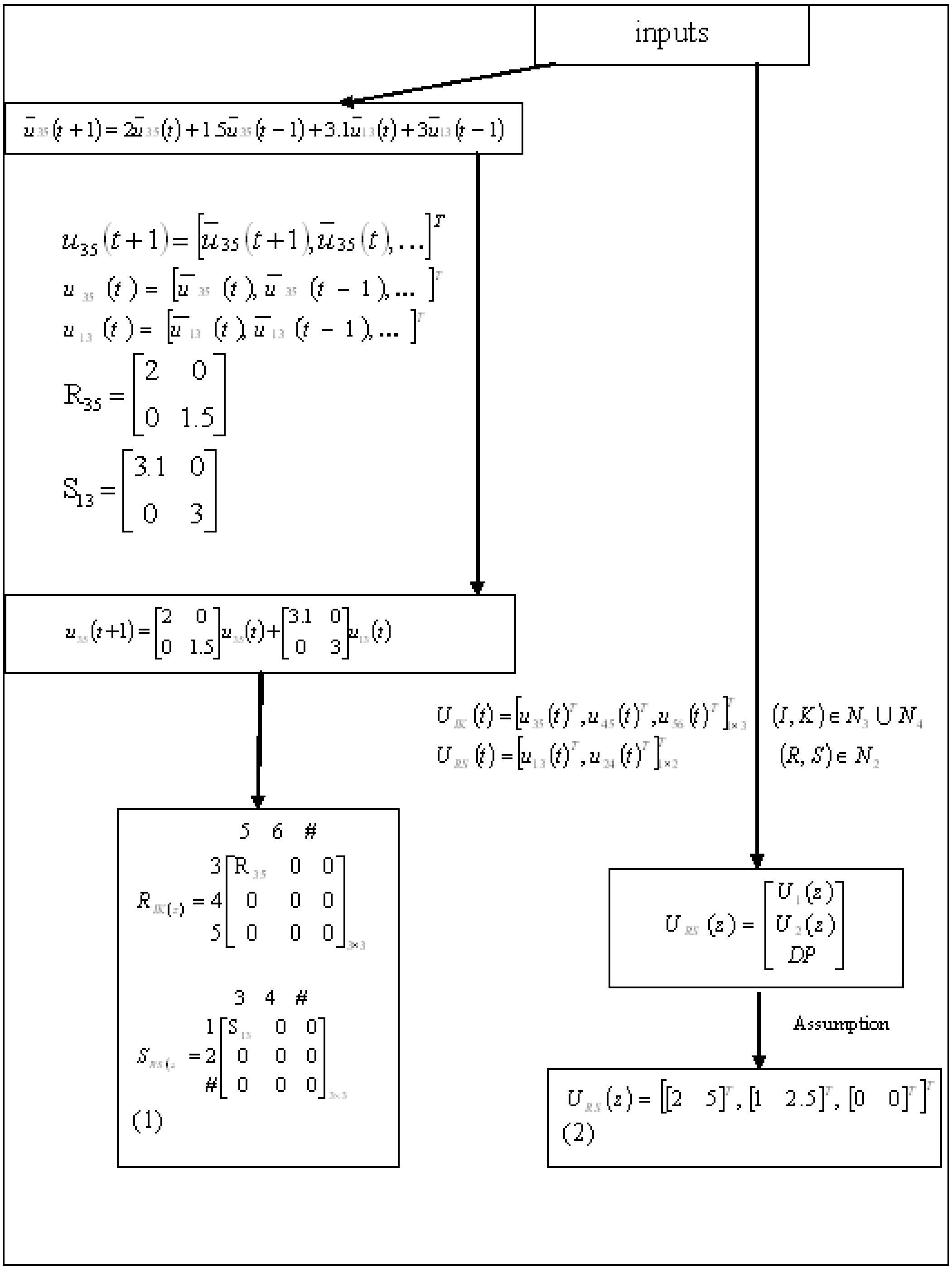

Figure 8. Inputs of numerical example analysis

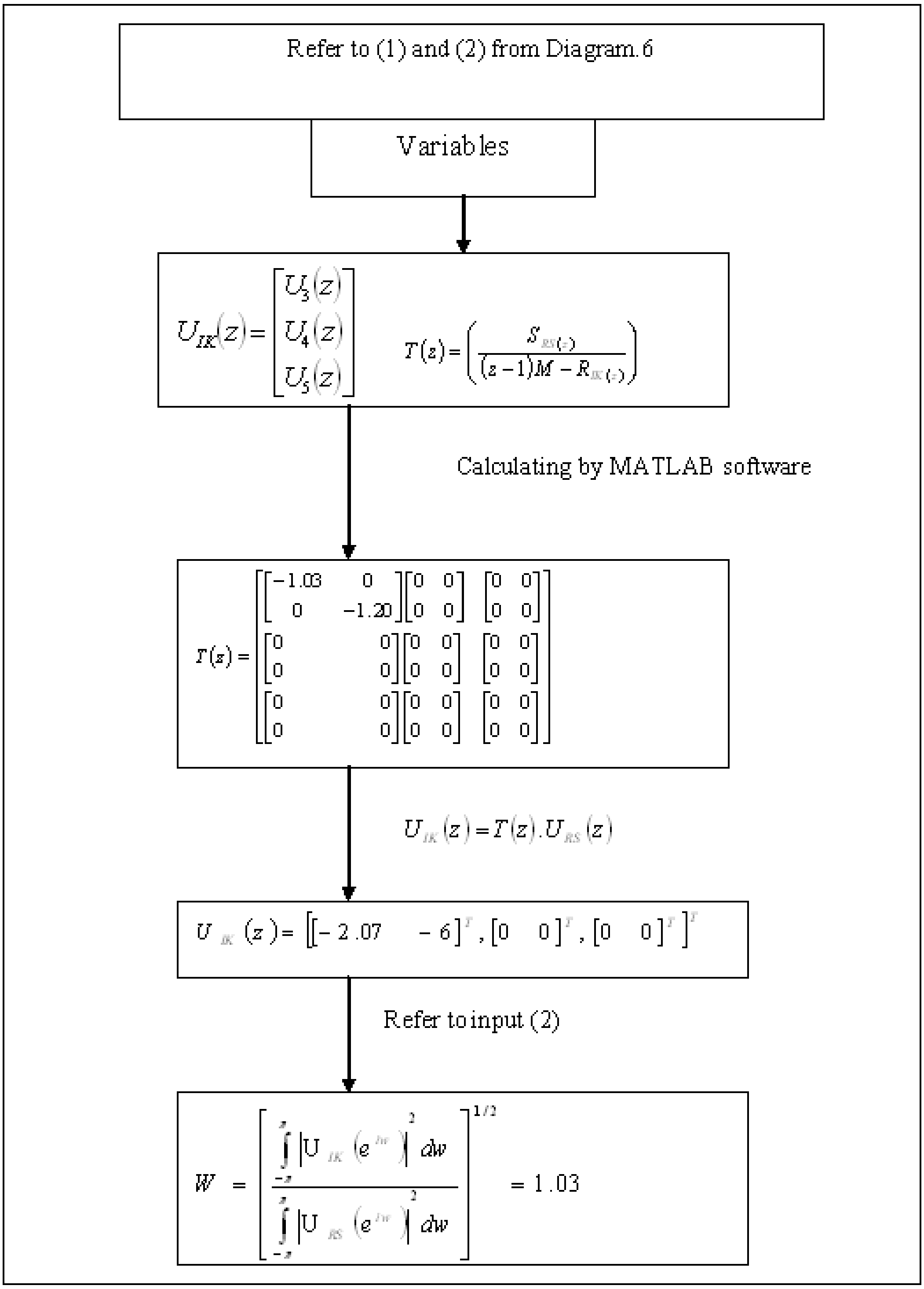

Figure 9. Measuring BWE of assumed supply network topology

To evaluate the efficiency of the proposed approach, the results of the proposed model are compared with results of the existing models in the literature. Table 1 compares results of our model and Ouyang and Li (2010) approach with the results of the calculating BWE.

Table1. Comparison between methodologies

Research	Method for BWE	Formula for calculating BWE	Result
Ouyang and Li (2010) Ouyang and Li (2010)	Plotting and determine BWE	$Sup\left\|T_v(e^{iw})\right\|_\infty > 1$	g(z)>0 Supply network experience bullwhip effect
Our method	Exact Formula with Fourier transform	$W = \left[\dfrac{\int_{-\pi}^{\pi}\left\|\mathrm{U}_{IK}\left(e^{iw}\right)\right\|^2 dw}{\int_{-\pi}^{\pi}\left\|\mathrm{U}_{RS}\left(e^{iw}\right)\right\|^2 dw}\right]^{1/2}$	W=1.03>=1 Supply network experience bullwhip effect

The formula of this research turns out using a mathematical method called frequency domain analysis. The major target of this paper is analyzing bullwhip effect considering exogenous uncertainty in supply networks using Fourier transform in order to simplify the relevant calculations. Basedon Ouyang and Li (2010) approach, calculating bullwhip effect has a complex structure, so a simplification equation is used for calculating the BWE. By plotting all elements of this simplification as function of *w* check the peaks exceeds 1 and so the BWE is detected in this way. This methodology gives exact formula for calculating the BWE. Therefore This comparison shows that the approach which can derive exact solution with Fourier transform for calculation the BWE is superior to Ouyang and Li (2010) approach.

4. Conclusions and further research

This paper proposed a model for analyzing order sequences deviation and BWE in supply networks under exogenous uncertainty. Presenting supply network concept and formulating it as a Markovian chain, the paper has derived robust analytical conditions to present the system under Markovian uncertainties which are assumed to be exogenous to any member of supply network. The presented model provides a basis for developing exact formula for analyzing BWE in any single-product supply network topologies considering LTI inventory management policies and exogenous uncertainty. The mentioned formula is obtained by the mathematical method called FDA. The numerical example indicated how the presented framework enabled supply managers to study the effect of various factors (e.g. network structure) under robust conditions (exogenous uncertainty), on the BWE in supply network. The modeling framework and analysis results presented in this paper are applicable to supply networks with any general topology. So as the future research it can be applied in a real case and evaluating the result from real data**.** The model presented in this research can be further extended by focusing on endogenous uncertainty (controllable conditions by network suppliers, e.g. ordering policy). Considering variant lead-times in supply network and presenting non-linear ordering policies for determining system control framework can be other further researches. However, considering the effect of information sharing and exogenous uncertainty on the BWE magnitude based on shared supply network information can be suggested as further research.

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