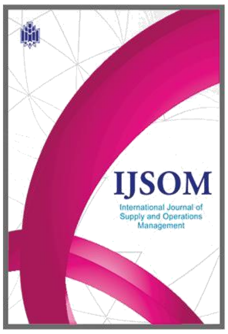
International Journal of Supply and Operations Management

IJSOM



November 2020, Volume 7, Issue 4, pp. 384-409
ISSN-Print: 2383-1359
ISSN-Online: 2383-2525
www.ijsom.com

A Single-Manufacturer Multi-Retailer Integrated Inventory Model with Price Dependent Demand and Stochastic Lead Time

Bibhas C. Giri [a], Anamika Dash[*,a], Ashis Kumar Sarkar [a]

[a] *Department of Mathematics Jadavpur University Kolkata, India*

Abstract

This paper considers a two-level supply chain which is composed of a single manufacturer and multiple retailers. The ordered quantities of the retailers are delivered in some equal sized batches from the manufacturer. Customer demand is linearly dependent on the retail price of the product. Lead time is random and it follows a normal distribution. The proposed models are developed for both the centralized and the decentralized scenarios. In the decentralized model, a Stackelberg gaming approach is used to find the optimal solution. The developed models are illustrated by a numerical example. It is found that integration of the supply chain players gives an impressive increment in profit of the whole supply chain. Sensitivity analysis is also carried out to explore the impacts of key parameters on the expected average profit of the supply chain system.

Keywords: Two-level supply chain; single manufacturer; multiple retailers; price dependent demand; stochastic lead time.

1. Introduction

The supply chain management involves with the activities for coordinating raw materials, information, production and financial flow to fulfil the customer demand with the aim of maximizing customer value and gaining competitive advantage in the market. In addition to efficient business management, supply chain spans various fields such as livestock management, waste management. Nowadays, supply chain plays an important role even in disaster management also (Khalilpourazari et al., 2020). In a supply chain, raw materials are delivered to the manufacturer from the supplier. Then produced items are transferred from the manufacturer to the retailer and finally delivered to the end customers to meet their demands. The most important factor that affects the consumer demand is the retail price of the product. The customer demand falls with the increase in retail price. So, many companies are now focusing on achieving the best pricing strategy to increase the sales volume. Many researchers (Ray and Jewkes, 2004; Banerjee, 2005; Yang et al., 2009) have also developed their models considering price-sensitive demand.

Nowadays, due to the globalization of the marketplace and rapidly increasing competition between various organisations, companies are facing many obstacles to compete exclusively. So, collaboration between different business units leads to a significant way to obtain spirited advantage. For better efficiency, the supply chain players are showing great interest in making their decisions jointly. Goyal (1976) was the first author who introduced integration between supply chain members in inventory model with finite production rate. Later, his work has encouraged many researchers (Banerjee, 1986; Goyal and Nebebe, 2000; Ben-Daya and Hariga, 2004; Pandey et al., 2007; Glock, 2012) to develop various integrated inventory models.

* Corresponding author email address: anamika1994.dash@gmail.com

Lead time refers to the time interval between order placement and receiving the delivery of items. It may be random in nature. It affects the demand forecasting and makes the customer looking for alternatives. So, for an efficient management of supply chain and facilitate of economic success, it is essential to concentrate on such a sensible factor. The literary review reflects that numerous research works have been carried out focusing on lead time (Ouyang et al., 2004; Lee, 2007). In two-level supply chain system, an interesting strategic problem arises at the retailer's end in determining optimal ordering quantity and the number of shipments decided by the manufacturer to transfer the ordered quantity of the retailer. Keeping in mind the importance of lead time and focusing on multi-retailer scenario, in our model, we develop a single-manufacturer multi-retailer supply chain model assuming that the lead time is a random variable following normal distribution and the customer demand is influenced by its selling price.

2. Literature Review

Over the last few decades, integration of vendor and buyer in supply chain management has attracted the attention of researchers remarkably. The key objective of their researches is to enhance the efficiency of the supply chain through the cooperation of supply chain players. In this section, we review some existing relevant literatures on supply chain models under several assumptions spanning across the streams – price-sensitive demand, stochastic lead time and multi-retailer inventory system.

Determination of the best pricing strategy and inventory decisions that influence the customer demand has been focused by many researchers and practitioners. A seminal research work in this direction was done initially by Whitin (1955). Later, many researchers (Yang et al., 2009; Khanra et al., 2010; Maragatham and Lakshmidevi, 2014) investigated joint pricing and inventory problems. Sohrabi et al. (2015) investigated a supplier selection problem in the vendor managed inventory (VMI) scenario under price dependent customer demand. Alfares and Ghaithan (2016) developed an inventory model considering storage time-dependent holding cost, quantity discount and price-dependent demand. Giri et al. (2017) implemented a profit sharing contract between the retailer and the manufacturer under price-sensitive market demand. Maragatham and Palani (2017) determined an optimal replenishment policy aimed at minimization of the total cost under time dependent ordering and holding costs and deterioration rate with price sensitive demand. Rastogi et al. (2018) derived optimal retail price and optimal time interval for an economic order quantity (EOQ) model considering non-instantaneous deterioration of items and price-sensitive consumer demand. Pervin et al. (2019) developed a two-level supply chain model with deteriorating multiple items under trade-credit policy and price- and stock-dependent demand. Biswas et al. (2019) developed an integrated inventory model and derived optimal decisions in order to optimize the system profitability for deteriorating item and price-dependent demand rate. Pervin et al. (2019) incorporated inflation in an economic production quantity (EPQ) inventory model for deteriorating items with preservation technology investment under price- and stock-dependent demand. Sarkar et al.(2019) proposed a price discount policy where the vendor offers a price discount depending on the buyer's ordered quantity in a price-sensitive market. Mashud et al.(2019) studied a deteriorating inventory model with selling price dependent demand under trade credit policy and preservation technology investment. Das et al.(2020) applied preservation technology in an inventory model for non-instantaneous deteriorating items, price sensitive demand and partially backlogged shortage. Mashud et al.(2020) implemented both the advance payment and delay payment in a supply chain model with the trade credit policy under price- and advertisement-sensitive customer demand.

Lead time has attracted the interest of many researchers in the areas of inventory management and production control. It plays a crucial role in business management. Lead time variability often causes shortages and, to hold the market demand, sometimes the retailers offer price discount or any other attractive offers on backorder demand (Roy et al., 2018). Lead time can be shortened by paying an additional cost. Many researchers (Ouyang et al., 2007; Ho, 2009; Glock, 2012; Vijayashree and Uthayakumar, 2017) investigated several production-inventory models with lead time reduction. Li et al. (2011) considered controllable lead time in their models. Sajadieh et al. (2009) allowed the uncertainty of the lead time with exponential distribution in a vendor-buyer model with equal batch shipments. Later, Hoque (2013) extended this model by considering combined equal and unequal batch shipments and showed that the model with normal distribution is more profitable than the model with exponential distribution for lead time. Vijayashree and Uthayakumar (2015) provided an effective iterative algorithm to minimize the cost of an integrated inventory model under controllable lead time. Lin (2016) derived production strategy and investment policy for reducing lead time variability in a vendor-buyer supply chain model with stochastic lead time. Giri et al. (2018) developed a closed-loop supply chain model considering learning effect in production in a stochastic lead time scenario under price and quality dependent demand. Wangsa and Wee (2019) studied an integrated inventory model with inspection error, stochastic lead time and freight cost. Then concluded that type I error is more significant than type II error and the distance between buyer and vendor impacts the lead time and the total cost effectively.

In today's competitive business world, it is hard to find that one manufacturer sells a product to a single retailer. It is more practical to focus on the scenarios where the manufacturer supplies the product to several retailers or trading for multiple items. Khalilpourazari et al. (2019a) investigated an economic order quantity model for multiple imperfect items. Khalilpourazari et al. (2019b) developed an EPQ model for multiple defective items with random defective rate and rework process. Khalilpourazari et al. (2019c) considered a multi-item EOQ model with partially backlogging shortages and warehouse capacity constraint. Researchers are also currently showing a great interest in developing such multi-retailer models. Taleizadeh et al. (2012) compared the results of the PSO (Particle Swarm Optimization) approach and GA (Genetic Algorithm) approach in a multi-product inventory model comprised of a single vendor and multiple buyers where the lead time varies with the lot size. Jha and Shanker (2013) considered a single-vendor multi-buyer supply chain model under controllable lead time and applied the Lagrangian multiplier technique to evaluate optimal results under service level constraints. Poorbagheri and Niaki (2014) studied a VMI model composed of a single vendor and multiple retailers under stochastic demand. Giri and Roy (2015) developed a single-vendor multi-buyer inventory model considering controllable lead time to maximize the profit of the whole supply chain. Uthayakumar et al. (2019) studied a single-vendor multi-buyer integrated model with stochastic demand and controllable lead time.

The above literature review reveals that researchers have studied single-manufacturer multi-retailer models considering important factors like stochastic lead time, price dependent market demand, etc. separately. A single-manufacturer multi-retailer supply chain model has not been studied where the customer demand is price dependent and the lead time is stochastic. To reflect a real business scenario, it is worth to incorporate these key factors under one umbrella. Inspired by the previous research works, in this paper, our aim is to fulfill this gap and answer the following research questions.

- What will be the optimal pricing and the batch shipment policy for the retailers and the manufacturer in a price sensitive market ?
- How does the stochastic nature of lead time influence optimal inventory decisions in a single manufacturer multi-retailer model with price-dependent demand ?
- What is the impact of integration on the profitability and managerial decisions of the supply chain ?

A comparison of the current work with the relevant existing literature is given in Table 1.

Table 1. Comparison of the present paper with the existing literature

Authors	Model type	Batch shipment	Demand	Lead time
Sajadieh et al.(2009)	Single-manufacturer single-retailer	Equal	Deterministic	Stochastic
Jha et al. (2013)	Single-manufacturer multi -retailer	Equal	Deterministic	Controllable
Hoque(2013)	Single-manufacturer single-retailer	Equal and unequal	Deterministic	Stochastic
Yang et al. (2013)	Single-manufacturer multi-retailer	No	Price- dependent	No
Mandal & Giri (2015)	Single-manufacturer multi-retailer	Equal	Stochastic	Controllable
Giri & Masanta (2019)	Single-manufacturer single-retailer	Equal	Deterministic	Stochastic
This paper	Single-manufacturer multi-retailer	Equal	Price- dependent	Stochastic

Remaining of this paper is summarized as follows: Section 3 provides assumptions and notations we have used throughout the paper. In Section 4, mathematical models are formulated and solution procedures are given to derive the optimal solutions. A numerical example is explained in Section 5 to analyze the proposed models. In section 6, sensitivity analysis with respect to some key parameters is carried out. Lastly, the conclusion is drawn and future research directions are provided in Section 7.

3. Notations and Assumptions

Notations used for developing the proposed models are given below. The subscript ‘ i ’ is used to indicate the i-th retailer.

Index:

i	Index of retailer, $i = 1,2,\ldots N$

Parameters:

R	Production rate
A_v	Set up cost /set up
h_v	Holding cost / item / unit time ($h_v > h_i$)

F	Transportation cost per batch shipment
w	Unit wholesale price
N	Number of retailers
Q	Total order quantity [$= \sum_{i=1}^{N} Q_i$]
D	Total market demand [$= \sum_{i=1}^{N} D_i$]
EAP_V	Average expected profit of the manufacturer
L	Lead time, a random variable
A_i	Ordering cost of the i-th retailer / order
h_i	Holding cost of the i-th retailer / item / unit time
D_i	Demand rate of the i-th retailer [$R > \sum_{i=1}^{N} D_i$]
a_i	Basic market demand of the i-th retailer
β_i	Consumer sensitivity coefficient to retail price of the i-th retailer
Q_i	Ordering quantity of the i-th retailer
c_i	Shortage cost of the i-th retailer / item / unit time
r_i	Reorder point of the i-th retailer
l_i	Lead time variable of the i-th retailer
T_i	Cycle time of the i-th retailer
σ_i	Standard deviation of the lead time of the i-th retailer
$f_L(.)$	Probability density function of the lead time
EAP_i	Average expected profit of the i-th retailer

Decision variables:

n	Number of batches delivered to each retailer
z_i	Batch size of the i-th retailer
p_i	Unit retail price of the i-th retailer

The basic assumptions made to formulate the proposed models are as follows:

1. The manufacturer produces a single product and meets the demand of multiple retailers.
2. The i-th retailer places his order of quantity Q_i. The manufacturer produces the total order quantity $\sum_{i=1}^{N} Q_i$ of all retailers in one set up at a production rate R, and then transfers the ordered quantity of the i-th retailer in n equal batches of size z_i such that $\frac{Q_i}{D_i} = \frac{Q}{D}$ for all $i = 1,2,....,N$.
3. The consumer's demand rate depends linearly on the selling price of the product, i.e. the consumer demand rate at the i-th retailer is $D_i(p_i) = a_i - \beta_i p_i$, where a_i represents the basic consumer demand and β_i is a positive integer such that $a_i > \beta_i p_i$ for all $i = 1,2,....,N$.
4. The production rate is constant and higher than the collective demand rates of all the retailers i.e. $R > \sum_{i=1}^{N} D_i$.
5. Shortages are allowed and are assumed to be completely backlogged at each retailer's end.
6. The i-th retailer places his next order when its stock level reaches a certain reorder point r_i.
7. The lead time is stochastic in nature and follows a normal distribution, and the lead time for all shipments are independent of each other.

4. Model Development

We assume that, the manufacturer transfers the order quantity Q_i of the i-th retailer in n equal batch shipments of size z_i. If the total order quantity of N retailers is Q, then we have $Q_i = nz_i$ and $Q = \sum_{i=1}^{N} Q_i$.

The i-th retailer places the next order when the stock reaches to a level r_i. The reorder level is determined so as to arrive the shipment to the retailer's end at or before the time of selling this r_i quantity at a demand rate D_i. The mean lead time is $\frac{r_i}{D_i}$. Due to various reasons, the batches may reach the retailer's end early or late. So, depending on the duration of lead time three cases may arise:

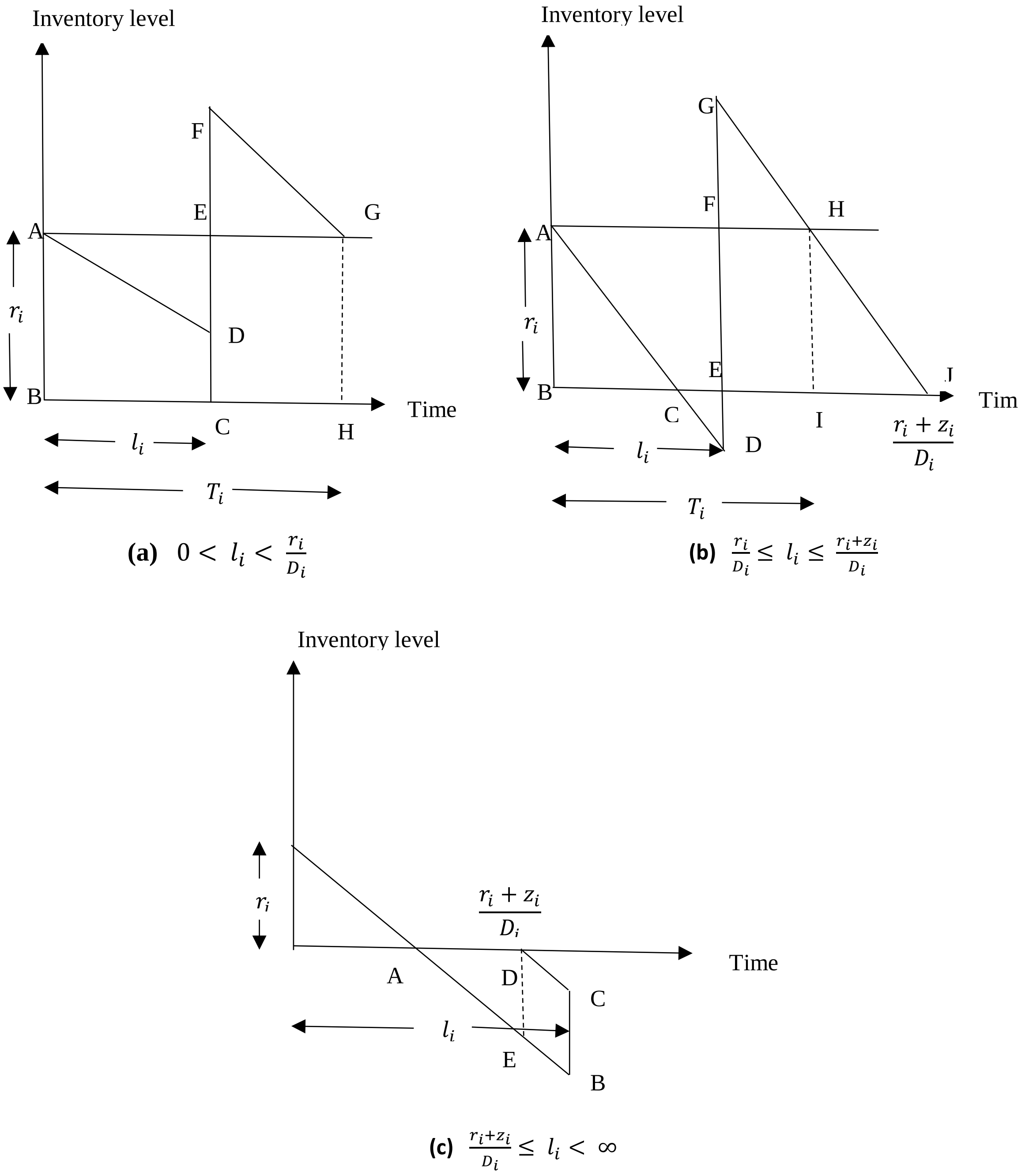


Figure 1. Inventory of i- th retailer under stochastic lead time

Case(i) *When the batch z_i reaches to the retailer earlier i.e.* $0 < l_i < \frac{r_i}{D_i}$.

In this case, the holding inventory area of the i-th retailer is determined from Figure 1(a) as

$$= \text{Area}(\triangle ABCD + \square CEGH + \triangle EFG)$$

$$= \left[\frac{1}{2} \times (AB + CD) \times BC\right] + (CE \times EG) + \left(\frac{1}{2} \times EF \times EG\right)$$

$$= \frac{1}{2}\left(r_i + r_i - D_i l_i\right) l_i + \frac{r_i\left(z_i - D_i l_i\right)}{D_i} + \frac{1}{2}\left(r_i - D_i l_i + z_i - r_i\right).\frac{\left(z_i - D_i l_i\right)}{D_i}$$

$$= \frac{1}{2}\left[\frac{z_i^2}{D_i} + 2z_i\left(\frac{r_i}{D_i} - l_i\right)\right] \text{ where } r_i = \frac{z_i D_i}{R}$$

The ordered quantity of the i-th retailer Q_i for n shipments is given by

$$\frac{n}{2}\left[\frac{z_i^2}{D_i} + 2z_i\left(\frac{r_i}{D_i} - l_i\right)\right]$$

The expected inventory holding cost for the ordered quantity Q_i of the i-th retailer is

$$nh_i \int_0^{\frac{r_i}{D_i}} \frac{1}{2}\left[\frac{z_i^2}{D_i} + 2z_i\left(\frac{r_i}{D_i} - l_i\right)\right] f_L(l_i) dl_i$$

Case (ii) *When the batch z_i reaches late to the i-th retailer and the lead time l_i lies in the range* $\frac{r_i}{D_i} \le l_i \le \frac{r_i + z_i}{D_i}$.

In this case, both inventory and shortages occur at the retailer's end. From Figure 1(b), shortage area of the i-th retailer is obtained as Area ($\triangle$ CDE)

$$= \frac{\left(D_i l_i - r_i\right)^2}{2D_i}$$

So, the expected shortage cost at the i-th retailer for n batches is given by $nc_i \int_{\frac{r_i}{D_i}}^{\frac{r_i+z_i}{D_i}} \frac{\left(D_i l_i - r_i\right)^2}{2D_i} f_L(l_i) dl_i$

Holding inventory area of the i-th retailer for the batch z_i is

$$= \text{Area}(\triangle ABC + \square EFHI + \triangle FGH)$$

$$= \frac{r_i^2}{2D_i} + \frac{r_i\left(z_i - D_i l_i\right)}{D_i} + \frac{\left(z_i - D_i l_i\right)^2}{2D_i}$$

$$= \frac{\left(z_i - D_i l_i + r_i\right)^2}{2D_i}$$

Hence the expected holding cost at the i-th retailer for n shipments is obtained as

$$nh_i \int_{\frac{r_i}{D_i}}^{\frac{r_i+z_i}{D_i}} \frac{\left(z_i - D_i l_i + r_i\right)^2}{2D_i} f_L(l_i) dl_i$$

It is assumed that, during this delay period, the batches remain in the manufacturer's stock house. So it causes an extra holding cost to the manufacturer. The extra inventory for this delayed delivery is

$$\sum_{i=1}^{N} \frac{nz_i\left(D_i l_i - r_i\right)}{D_i}.$$

Therefore, in this scenario, the extra holding cost paid by the manufacturer is

$$h_v \sum_{i=1}^{N} \int_{\frac{r_i}{D_i}}^{\frac{r_i+z_i}{D_i}} \frac{nz_i\left(D_i l_i - r_i\right)}{D_i} f_L\left(l_i\right) dl_i$$

Case (iii) *When the batch z_i reaches late to the retailer and lead time lies in the range* $\frac{r_i+z_i}{D_i} \le l_i < \infty$.

In this case, only shortages occur at the retailer's end and, from Figure 1(c), the shortage area for the batch z_i is obtained as = Area ($\triangle$ABCD)

$$= \frac{z_i^2}{2D_i} + \text{Area } (\square \text{BCDE})$$

$$= \frac{z_i^2}{2D_i} + z_i\left(l_i - \frac{z_i + r_i}{D_i}\right)$$

So, the expected shortage cost of the i-th retailer for all batch shipments is

$$nc_i \int_{\frac{r_i+z_i}{D_i}}^{\infty} \left[\frac{z_i^2}{2D_i} + z_i\left(\frac{D_i l_i - z_i - r_i}{D_i}\right)\right] f_L(l_i) dl_i$$

Similarly as case(ii), the additional expected holding cost of the manufacturer is

$$h_v \sum_{i=1}^{N} \int_{\frac{r_i+z_i}{D_i}}^{\infty} \frac{nz_i\left(D_i l_i - r_i\right)}{D_i} f_L\left(l_i\right) dl_i$$

Combining all three cases, the expected holding cost at the i-th retailer for all batch shipments is given by

$$nh_i\left[\int_0^{\frac{r_i}{D_i}} \frac{1}{2}\left[\frac{z_i^2}{D_i} + 2z_i\left(\frac{r_i}{D_i} - l_i\right)\right] f_L(l_i) dl_i + \int_{\frac{r_i}{D_i}}^{\frac{r_i+z_i}{D_i}} \frac{\left(z_i - D_i l_i + r_i\right)^2}{2D_i} f_L(l_i) dl_i\right]$$

and the expected shortage cost for all batch shipments is

$$nc_i\left[\int_{\frac{r_i}{D_i}}^{\frac{r_i+z_i}{D_i}} \frac{\left(D_i l_i - r_i\right)^2}{2D_i} f_L(l_i) dl_i + \int_{\frac{r_i+z_i}{D_i}}^{\infty} \left[\frac{z_i^2}{2D_i} + z_i\left(\frac{D_i l_i - z_i - r_i}{D_i}\right)\right] f_L(l_i) dl_i\right]$$

4.1. Decentralized Model

In the decentralized model scenario, the manufacturer and the retailers make their decisions independently in order to improve their own profits. Here we incorporate a Stackelberg gaming structure where the retailers act as the leader and the manufacturer is the follower. The manufacturer sets the number of shipments. Then, taking this response function into consideration, the retailers determine optimal retail prices of the product and batch sizes.

Retailer's profit function:

The expected total profit of the i-th retailer is

$$p_iQ_i - wQ_i - A_i - nF - nh_i\left[\int_0^{\frac{r_i}{D_i}} \frac{1}{2}\left[\frac{z_i^2}{D_i} + 2z_i\left(\frac{r_i}{D_i} - l_i\right)\right] f_L(l_i)dl_i + \int_{\frac{r_i}{D_i}}^{\frac{r_i+z_i}{D_i}} \frac{\left(z_i - D_il_i + r_i\right)^2}{2D_i} f_L(l_i)dl_i\right]$$
$$-nc_i\left[\int_{\frac{r_i}{D_i}}^{\frac{r_i+z_i}{D_i}} \frac{\left(D_il_i - r_i\right)^2}{2D_i} f_L(l_i)dl_i + \int_{\frac{r_i+z_i}{D_i}}^{\infty} \left[\frac{z_i^2}{2D_i} + z_i\left(\frac{D_il_i - z_i - r_i}{D_i}\right)\right] f_L(l_i)dl_i\right]$$

The average expected profit of the i-th retailer is obtained as

$$EAP_i\left(z_i, p_i\right) = p_iD_i - wD_i - \frac{(A_i + nF)D_i}{Q_i} - nh_i\left[\int_0^{\frac{r_i}{D_i}} \frac{1}{2}\left[\frac{z_i^2}{Q_i} + \frac{2}{n}\left(r_i - D_il_i\right)\right] f_L(l_i)dl_i + \int_{\frac{r_i}{D_i}}^{\frac{r_i+z_i}{D_i}} \frac{\left(z_i - D_il_i + r_i\right)^2}{2Q_i} f_L(l_i)dl_i\right]$$
$$-nc_i\left[\int_{\frac{r_i}{D_i}}^{\frac{r_i+z_i}{D_i}} \frac{\left(D_il_i - r_i\right)^2}{2Q_i} f_L(l_i)dl_i + \int_{\frac{r_i+z_i}{D_i}}^{\infty} \left[\frac{z_i^2}{2Q_i} + z_i\left(\frac{D_il_i - z_i - r_i}{Q_i}\right)\right] f_L(l_i)dl_i\right] \tag{1}$$

Proposition 1. The average expected profit function of the i-th retailer is concave in z_i for given p_i provided that

$$2(A_i + nF)D_i + n\left(h_i + c_i\right) \int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} D_i^2 l_i^2 f_L(l_i)dl_i > 0 \cdot \tag{2}$$

Proof. Differentiating (1) with respect to z_i, we obtain

$$\frac{\partial EAP_i}{\partial z_i} = \frac{\partial}{\partial z_i}\left[-\frac{(A_i + nF)D_i}{nz_i}\right] - \frac{\partial}{\partial z_i}\left[\frac{h_i}{2}\int_0^{\frac{r_i}{D_i}} \left[z_i + 2\left(r_i - D_il_i\right)\right] f_L(l_i)dl_i\right] - \frac{\partial}{\partial z_i}\left[\frac{h_i}{2}\int_{\frac{r_i}{D_i}}^{\frac{r_i+z_i}{D_i}} \frac{\left(z_i - D_il_i + r_i\right)^2}{z_i} f_L(l_i)dl_i\right]$$
$$-\frac{\partial}{\partial z_i}\left[\frac{c_i}{2}\int_{\frac{r_i}{D_i}}^{\frac{r_i+z_i}{D_i}} \frac{\left(D_il_i - r_i\right)^2}{z_i} f_L(l_i)dl_i\right] - \frac{\partial}{\partial z_i}\left[\frac{c_i}{2}\int_{\frac{r_i+z_i}{D_i}}^{\infty} \left[z_i + 2\left(D_il_i - z_i - r_i\right)\right] f_L(l_i)dl_i\right]$$

Now, using $r_i = \frac{z_iD_i}{R}$, the above equation reduces as

$$\frac{\partial EAP_i}{\partial z_i} = \frac{\partial}{\partial z_i}\left[-\frac{(A_i + nF)D_i}{nz_i}\right] - \frac{\partial}{\partial z_i}\left[\frac{h_i}{2}\int_0^{\frac{z_i}{R}} \left[z_i + 2\left(\frac{z_iD_i}{R} - D_il_i\right)\right] f_L(l_i)dl_i\right] - \frac{\partial}{\partial z_i}\left[\frac{h_i}{2}\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{\left(z_i - D_il_i + \frac{z_iD_i}{R}\right)^2}{z_i} f_L(l_i)dl_i\right]$$
$$-\frac{\partial}{\partial z_i}\left[\frac{c_i}{2}\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{\left(D_il_i - \frac{z_iD_i}{R}\right)^2}{z_i} f_L(l_i)dl_i\right] - \frac{\partial}{\partial z_i}\left[\frac{c_i}{2}\int_{\frac{z_i(R+D_i)}{RD_i}}^{\infty} \left[z_i + 2\left(D_il_i - z_i - \frac{z_iD_i}{R}\right)\right] f_L(l_i)dl_i\right]$$

$$=\frac{(A_i+nF)D_i}{nz_i^2}-\frac{h_i}{2}\left[\int_0^{\frac{z_i}{R}}\frac{\partial}{\partial z_i}\left[z_i+2\left(\frac{z_iD_i}{R}-D_il_i\right)\right]f_L(l_i)dl_i+\left(z_i+2\left(\frac{z_iD_i}{R}-D_i.\frac{z_i}{R}\right)\right).f_L\left(\frac{z_i}{R}\right)\frac{\partial}{\partial z_i}\left(\frac{z_i}{R}\right)\right]$$

$$-\frac{h_i}{2}\left[\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}}\frac{\partial}{\partial z_i}\left[\frac{\left(z_i-D_il_i+\frac{z_iD_i}{R}\right)^2}{z_i}\right]f_L(l_i)dl_i+\frac{\left(z_i-D_i.\frac{z_i(R+D_i)}{RD_i}+\frac{z_iD_i}{R}\right)^2}{z_i}f_L\left(\frac{z_i(R+D_i)}{RD_i}\right)\frac{\partial}{\partial z_i}\left(\frac{z_i(R+D_i)}{RD_i}\right)-\frac{\left(z_i-D_i.\frac{z_i}{R}+\frac{z_iD_i}{R}\right)^2}{z_i}f_L\left(\frac{z_i}{R}\right)\frac{\partial}{\partial z_i}\left(\frac{z_i}{R}\right)\right]$$

$$-\frac{c_i}{2}\left[\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}}\frac{\partial}{\partial z_i}\left[\frac{\left(D_il_i-\frac{z_iD_i}{R}\right)^2}{z_i}\right]f_L(l_i)dl_i+\frac{\left(D_i.\frac{z_i(R+D_i)}{RD_i}-\frac{z_iD_i}{R}\right)^2}{z_i}.f_L\left(\frac{z_i(R+D_i)}{RD_i}\right)\frac{\partial}{\partial z_i}\left(\frac{z_i(R+D_i)}{RD_i}\right)-\frac{\left(D_i.\frac{z_i}{R}-\frac{z_iD_i}{R}\right)^2}{z_i}.f_L\left(\frac{z_i}{R}\right)\frac{\partial}{\partial z_i}\left(\frac{z_i}{R}\right)\right]$$

$$-\frac{c_i}{2}\left[\int_{\frac{z_i(R+D_i)}{RD_i}}^{\infty}\frac{\partial}{\partial z_i}\left[z_i+2\left(D_il_i-z_i-\frac{z_iD_i}{R}\right)\right]f_L(l_i)dl_i-\left[z_i+2\left(D_i.\frac{z_i(R+D_i)}{RD_i}-z_i-\frac{z_iD_i}{R}\right)\right]f_L\left(\frac{z_i(R+D_i)}{RD_i}\right)\frac{\partial}{\partial z_i}\left(\frac{z_i(R+D_i)}{RD_i}\right)\right]$$

$$=\frac{(A_i+nF)D_i}{nz_i^2}-\frac{h_i}{2}\int_0^{\frac{z_i}{R}}\left(\frac{R+2D_i}{R}\right)f_L(l_i)dl_i-\frac{h_i}{2}\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}}\left(\frac{z_i^2(R+D_i)^2-R^2D_i^2l_i^2}{z_i^2R^2}\right)f_L(l_i)dl_i$$

$$+\frac{c_i}{2}\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}}\left(\frac{R^2D_i^2l_i^2-z_i^2D_i^2}{z_i^2R^2}\right)f_L(l_i)dl_i+\frac{c_i}{2}\int_{\frac{z_i(R+D_i)}{RD_i}}^{\infty}\left(\frac{R+2D_i}{R}\right)f_L(l_i)dl_i$$

(3)

$$\frac{\partial^2 EAP_i}{\partial z_i^2}=-\frac{2(A_i+nF)D_i}{nz_i^3}-\frac{h_i}{2}\left[\int_0^{\frac{z_i}{R}}\frac{\partial}{\partial z_i}\left(\frac{R+2D_i}{R}\right)f_L(l_i)dl_i+\left(\frac{R+2D_i}{R}\right)f_L\left(\frac{z_i}{R}\right)\frac{\partial}{\partial z_i}\left(\frac{z_i}{R}\right)\right]$$

$$-\frac{h_i}{2}\left[\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}}\frac{\partial}{\partial z_i}\left(\frac{z_i^2(R+D_i)^2-R^2D_i^2l_i^2}{z_i^2R^2}\right)f_L(l_i)dl_i+\left(\frac{z_i^2(R+D_i)^2-R^2D_i^2.\frac{z_i^2(R+D_i)^2}{R^2D_i^2}}{z_i^2R^2}\right)f_L\left(\frac{z_i(R+D_i)}{RD_i}\right)\frac{\partial}{\partial z_i}\left(\frac{z_i(R+D_i)}{RD_i}\right)-\left(\frac{z_i^2(R+D_i)^2-R^2D_i^2.\frac{z_i^2}{R^2}}{z_i^2R^2}\right)f_L\left(\frac{z_i}{R}\right)\frac{\partial}{\partial z_i}\left(\frac{z_i}{R}\right)\right]$$

$$+\frac{c_i}{2}\left[\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}}\frac{\partial}{\partial z_i}\left(\frac{R^2D_i^2l_i^2-z_i^2D_i^2}{z_i^2R^2}\right)f_L(l_i)dl_i+\left(\frac{R^2D_i^2.\frac{z_i^2(R+D_i)^2}{R^2D_i^2}-z_i^2D_i^2}{z_i^2R^2}\right).f_L\left(\frac{z_i(R+D_i)}{RD_i}\right)\frac{\partial}{\partial z_i}\left(\frac{z_i(R+D_i)}{RD_i}\right)-\left(\frac{R^2D_i^2.\frac{z_i^2}{R^2}-z_i^2D_i^2}{z_i^2R^2}\right).f_L\left(\frac{z_i}{R}\right)\frac{\partial}{\partial z_i}\left(\frac{z_i}{R}\right)\right]$$

$$+\frac{c_i}{2}\left[\int_{\frac{z_i(R+D_i)}{RD_i}}^{\infty}\frac{\partial}{\partial z_i}\left(\frac{R+2D_i}{R}\right)f_L(l_i)dl_i-\left(\frac{R+2D_i}{R}\right)f_L\left(\frac{z_i(R+D_i)}{RD_i}\right)\frac{\partial}{\partial z_i}\left(\frac{z_i(R+D_i)}{RD_i}\right)\right]$$

$$=-\frac{2(A_i+nF)D_i}{nz_i^{\,3}}-\frac{h_i}{2}\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}}\frac{2D_i^{\,2}l_i^{\,2}}{z_i^{\,3}}f_L(l_i)dl_i-\frac{c_i}{2}\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}}\frac{2D_i^{\,2}l_i^{\,2}}{z_i^{\,3}}f_L(l_i)dl_i \tag{4}$$

For given p_i, the average expected profit function of the i-th retailer is concave in z_i if $\frac{\partial^2 EAP_i}{\partial z_i^{\,2}}<0$ which implies that $2(A_i+nF)D_i+n\left(h_i+c_i\right)\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} D_i^{\,2}l_i^{\,2}f_L(l_i)dl_i>0\cdot$

Proposition 2. The average expected profit function of the i-th retailer is concave in p_i for given z_i if

$$2\beta_i+\left(h_i+c_i\right)\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}}\frac{\beta_i^{\,2}\left(z_i-Rl_i\right)^2}{z_iR^2}f_L(l_i)dl_i>0\cdot \tag{5}$$

Proof. After differentiating (1) with respect to p_i, we obtain

$$\frac{\partial EAP_i}{\partial p_i}=\frac{\partial}{\partial p_i}\left[p_iD_i-wD_i-\frac{(A_i+nF)D_i}{Q_i}\right]-\frac{\partial}{\partial p_i}\left[\frac{h_i}{2}\int_0^{\frac{r_i}{D_i}}\left[z_i+2\left(r_i-D_il_i\right)\right]f_L(l_i)dl_i\right]-\frac{\partial}{\partial p_i}\left[\frac{h_i}{2}\int_{\frac{r_i}{D_i}}^{\frac{r_i+z_i}{D_i}}\frac{\left(z_i-D_il_i+r_i\right)^2}{z_i}f_L(l_i)dl_i\right]$$
$$-\frac{\partial}{\partial p_i}\left[\frac{c_i}{2}\int_{\frac{r_i}{D_i}}^{\frac{r_i+z_i}{D_i}}\frac{\left(D_il_i-r_i\right)^2}{z_i}f_L(l_i)dl_i\right]-\frac{\partial}{\partial p_i}\left[\frac{c_i}{2}\int_{\frac{r_i+z_i}{D_i}}^{\infty}\left[z_i+2\left(D_il_i-z_i-r_i\right)\right]f_L(l_i)dl_i\right]$$

Now, using $r_i=\frac{z_iD_i}{R}$, the above equation reduces as

$$\frac{\partial EAP_i}{\partial p_i}=D_i+w\beta_i-\beta_ip_i+\frac{(A_i+nF)\beta_i}{Q_i}-\frac{\partial}{\partial p_i}\left[\frac{h_i}{2}\int_0^{\frac{z_i}{R}}\left[z_i+2\left(\frac{z_iD_i}{R}-D_il_i\right)\right]f_L(l_i)dl_i\right]-\frac{\partial}{\partial p_i}\left[\frac{h_i}{2}\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}}\frac{\left(z_i-D_il_i+\frac{z_iD_i}{R}\right)^2}{z_i}f_L(l_i)dl_i\right]$$
$$-\frac{\partial}{\partial p_i}\left[\frac{c_i}{2}\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}}\frac{\left(D_il_i-\frac{z_iD_i}{R}\right)^2}{z_i}f_L(l_i)dl_i\right]-\frac{\partial}{\partial p_i}\left[\frac{c_i}{2}\int_{\frac{z_i(R+D_i)}{RD_i}}^{\infty}\left[z_i+2\left(D_il_i-z_i-\frac{z_iD_i}{R}\right)\right]f_L(l_i)dl_i\right]$$

$$= D_i + w\beta_i - \beta_i p_i + \frac{(A_i+nF)\beta_i}{Q_i} - \frac{h_i}{2}\left[\int_0^{\frac{z_i}{R}} \frac{\partial}{\partial p_i}\left[z_i + 2\left(\frac{z_i D_i}{R} - D_i l_i\right)\right] f_L(l_i)dl_i + \left(z_i + 2\left(\frac{z_i D_i}{R} - D_i.\frac{z_i}{R}\right)\right).f_L\left(\frac{z_i}{R}\right)\frac{\partial}{\partial p_i}\left(\frac{z_i}{R}\right)\right]$$

$$-\frac{h_i}{2}\left[\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{\partial}{\partial p_i}\left[\frac{\left(z_i - D_i l_i + \frac{z_i D_i}{R}\right)^2}{z_i}\right] f_L(l_i)dl_i + \frac{\left(z_i - D_i.\frac{z_i(R+D_i)}{RD_i} + \frac{z_i D_i}{R}\right)^2}{z_i} f_L\left(\frac{z_i(R+D_i)}{RD_i}\right)\frac{\partial}{\partial p_i}\left(\frac{z_i(R+D_i)}{RD_i}\right) - \frac{\left(z_i - D_i.\frac{z_i}{R} + \frac{z_i D_i}{R}\right)^2}{z_i} f_L\left(\frac{z_i}{R}\right)\frac{\partial}{\partial p_i}\left(\frac{z_i}{R}\right)\right]$$

$$-\frac{c_i}{2}\left[\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{\partial}{\partial p_i}\left[\frac{\left(D_i l_i - \frac{z_i D_i}{R}\right)^2}{z_i}\right] f_L(l_i)dl_i + \frac{\left(D_i.\frac{z_i(R+D_i)}{RD_i} - \frac{z_i D_i}{R}\right)^2}{z_i}.f_L\left(\frac{z_i(R+D_i)}{RD_i}\right)\frac{\partial}{\partial p_i}\left(\frac{z_i(R+D_i)}{RD_i}\right) - \frac{\left(D_i.\frac{z_i}{R} - \frac{z_i D_i}{R}\right)^2}{z_i}.f_L\left(\frac{z_i}{R}\right)\frac{\partial}{\partial p_i}\left(\frac{z_i}{R}\right)\right]$$

$$-\frac{c_i}{2}\left[\int_{\frac{z_i(R+D_i)}{RD_i}}^{\infty} \frac{\partial}{\partial p_i}\left[z_i + 2\left(D_i l_i - z_i - \frac{z_i D_i}{R}\right)\right] f_L(l_i)dl_i - \left[z_i + 2\left(D_i.\frac{z_i(R+D_i)}{RD_i} - z_i - \frac{z_i D_i}{R}\right)\right] f_L\left(\frac{z_i(R+D_i)}{RD_i}\right)\frac{\partial}{\partial p_i}\left(\frac{z_i(R+D_i)}{RD_i}\right)\right]$$

$$= D_i + w\beta_i - \beta_i p_i + \frac{(A_i+nF)\beta_i}{Q_i} + h_i\int_0^{\frac{z_i}{R}} \frac{\beta_i(z_i - Rl_i)}{R} f_L(l_i)dl_i + h_i\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{\beta_i(z_i - Rl_i)\left[z_i R + D_i(z_i - Rl_i)\right]}{z_i R^2} f_L(l_i)dl_i$$

$$+c_i\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{D_i\beta_i(z_i - Rl_i)^2}{z_i R^2} f_L(l_i)dl_i - c_i\int_{\frac{z_i(R+D_i)}{RD_i}}^{\infty} \frac{\beta_i(z_i - Rl_i)}{R} f_L(l_i)dl_i \tag{6}$$

$$\frac{\partial^2 EAP_i}{\partial p_i^2} = h_i\left[\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{\partial}{\partial p_i}\left(\frac{\beta_i(z_i - Rl_i)\left[z_i R + D_i(z_i - Rl_i)\right]}{z_i R^2}\right) f_L(l_i)dl_i + \frac{\beta_i\left(z_i - R.\frac{z_i(R+D_i)}{RD_i}\right)\left[z_i R + D_i\left(z_i - R.\frac{z_i(R+D_i)}{RD_i}\right)\right]}{z_i R^2} f_L\left(\frac{z_i(R+D_i)}{RD_i}\right)\frac{\partial}{\partial p_i}\left(\frac{z_i(R+D_i)}{RD_i}\right)\right]$$

$$+c_i\left[\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{\partial}{\partial p_i}\left(\frac{D_i\beta_i(z_i - Rl_i)^2}{z_i R^2}\right) f_L(l_i)dl_i + \left(\frac{D_i\beta_i\left(z_i - R.\frac{z_i(R+D_i)}{RD_i}\right)^2}{z_i R^2}\right) f_L\left(\frac{z_i(R+D_i)}{RD_i}\right)\frac{\partial}{\partial p_i}\left(\frac{z_i(R+D_i)}{RD_i}\right)\right]$$

$$-c_i\left[\int_{\frac{z_i(R+D_i)}{RD_i}}^{\infty} \frac{\partial}{\partial p_i}\left(\frac{\beta_i(z_i - Rl_i)}{R}\right) f_L(l_i)dl_{ii} - \left(\frac{\beta_i\left(z_i - R.\frac{z_i(R+D_i)}{RD_i}\right)}{R}\right) f_L\left(\frac{z_i(R+D_i)}{RD_i}\right)\frac{\partial}{\partial p_i}\left(\frac{z_i(R+D_i)}{RD_i}\right)\right] - 2\beta_i$$

$$= -2\beta_i - (h_i + c_i)\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{\beta_i^2(z_i - Rl_i)^2}{z_i R^2} f_L(l_i)dl_i \tag{7}$$

Since $\beta_i > 0, h_i > 0$ and $c_i > 0$, it implies that $\frac{\partial^2 EAP_i}{\partial p_i^2} < 0$ if

$$2\beta_i + \left(h_i + c_i\right) \int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{\beta_i^2 \left(z_i - Rl_i\right)^2}{z_i R^2} f_L(l_i) dl_i > 0 \cdot$$

Manufacturer's profit function:
Total extra holding cost for the manufacturer, from both case (ii) and case (iii), is

$$h_v \sum_{i=1}^{N} \int_{\frac{r_i}{D_i}}^{\infty} \frac{nz_i \left(D_i l_i - r_i\right)}{D_i} f_L\left(l_i\right) dl_i$$

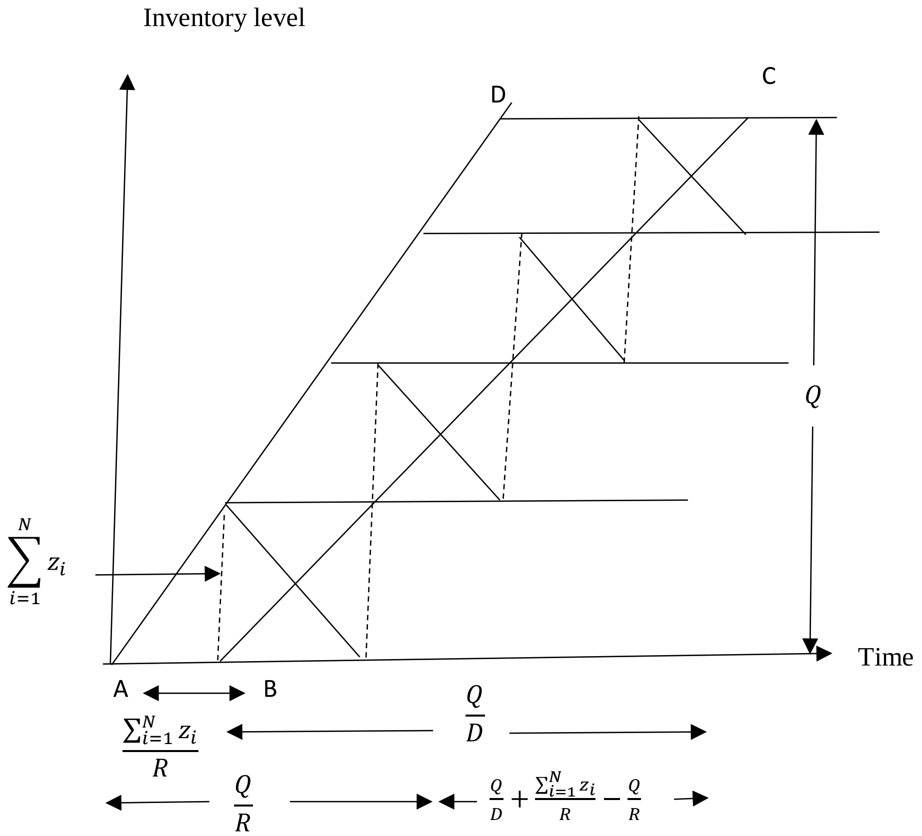


Figure 2. Joint inventory of the manufacturer and the retailers

In Figure 2, trapezium ABCD represents the joint inventory of the manufacturer-retailer system. Then, the average inventory of the system = Area (⏢ ABCD) $\times \frac{D}{Q}$

$$= \frac{1}{2} \times (AB + CD) \times Q \times \frac{D}{Q}$$

$$= \frac{1}{2}\left[\frac{\sum_{i=1}^{N} z_i}{R} + \left(\frac{Q}{D} + \frac{\sum_{i=1}^{N} z_i}{R} - \frac{Q}{R}\right)\right] D$$

$$= \frac{D\sum_{i=1}^{N} z_i}{R} + \frac{Q}{2}\left(1 - \frac{D}{R}\right)$$

The average holding area for N retailers $= \sum_{i=1}^{N}\left(\frac{z_i^2}{2D_i}\right)\left(\frac{D_i}{Q_i}\right) = \sum_{i=1}^{N}\frac{z_i^2}{2Q_i}$

The average inventory holding area of the manufacturer

= average joint inventory – average inventory of N retailers

$$= \frac{D\sum_{i=1}^{N} z_i}{R} + \frac{Q}{2}\left(1 - \frac{D}{R}\right) - \sum_{i=1}^{N}\frac{z_i^2}{2Q_i}$$

The average expected profit of the manufacturer is

$$EAP_v(n) = wD - \frac{A_v D}{Q} - h_v\left[\frac{D\sum_{i=1}^{N} z_i}{R} + \frac{Q}{2}\left(1 - \frac{D}{R}\right) - \sum_{i=1}^{N}\frac{z_i^2}{2Q_i}\right] - h_v\sum_{i=1}^{N}\int_{\frac{r_i}{D_i}}^{\infty}\left(D_i l_i - r_i\right)f_L\left(l_i\right)dl_i$$

$$= wD - \frac{A_v D}{Q} - h_v\left[\frac{D\sum_{i=1}^{N} z_i}{R} + \frac{Q}{2}\left(1 - \frac{D}{R}\right) - \sum_{i=1}^{N}\frac{z_i^2}{2Q_i}\right] - \sum_{i=1}^{N}\frac{h_v D_i \sigma_i}{\sqrt{2\pi}} \tag{8}$$

Proposition 3. The average expected profit function of the manufacturer is concave in n if $h_v s^2 < 2A_v D$. Then the optimal number of batch shipments is given by

$$n^* = \sqrt{\frac{R\left(2A_v D - h_v s^2\right)}{h_v s^2\left(R - D\right)}} \quad \text{where} \quad s = \sum_{i=1}^{N} z_i . \tag{9}$$

Proof. Assuming n as real, not just integer, we get from (8)

$$\frac{\partial EAP_v}{\partial n} = \frac{A_v D}{n^2 s} - \frac{h_v s}{2} + \frac{h_v s D}{2R} - \frac{h_v s}{2n^2} \tag{10}$$

$$\frac{\partial^2 EAP_v}{\partial n^2} = -\frac{2A_v D}{n^3 s} + \frac{h_v s}{n^3} \tag{11}$$

So, the manufacturer's average expected profit function EAP_V is concave in n if $\frac{\partial^2 EAP_v}{\partial n^2} < 0$. which implies that $h_v s^2 < 2A_v D$.

If the above condition holds, then by using the first order optimality condition i.e, solving $\frac{\partial EAP_v}{\partial n} = 0$ for n, we get the optimal number of shipments as $n = \sqrt{\frac{R\left(2A_v D - h_v s^2\right)}{h_v s^2\left(R - D\right)}}$.

Solution Algorithm

Taking the best response from the manufacturer, the average expected profit of the i-th retailer can be optimized using the following solution algorithm. Here, we first assign some initial values to the decision variables of the remaining retailers to maximize the average expected profit of one retailer.

Step 1: Set $k=1$.

Step 2: Set $i=1$ and $z_j = z_j^{(k\text{-}1)}$, $p_j = p_j^{(k\text{-}1)}$ for all $j = i+1, i+2, \ldots, N$.

Step 3: Optimize EAP_i taking n from the response function of the manufacturer and $z_j = z_j^{(k\text{-}1)}$, $p_j = p_j^{(k\text{-}1)}$ for all $j = i+1, i+2, \ldots, N$. Set the optimal results as $z_i = z_i^{(k)}$ and $p_i = p_i^{(k)}$.

Step 4: Set $i = i+1$.

Step 5: Optimize EAP_i taking n from the response function of the manufacturer and $z_j = z_j^{(k)}$, $p_j = p_j^{(k)}$ for $j = 1,2, \ldots, i-1$ and $z_j = z_j^{(k\text{-}1)}$, $p_j = p_j^{(k\text{-}1)}$ for $j = i+1$, $i+2, \ldots, N$. Set the optimal results as $z_i = z_i^{(k)}$ and $p_i = p_i^{(k)}$.

Step 6: Repeat steps 4 and 5 until $i = N$.

Step 7: Stop if $z_j^{(k)} = z_j^{(k\text{-}1)}$ and $p_j^{(k)} = p_j^{(k\text{-}1)}$ for all $j = 2, 3, \ldots, N$ and consider $z_j^{(*)} = z_j^{(k)}$ and $p_j^{(*)} = p_j^{(k)}$ for all $j = 1,2,3, \ldots, N$. Otherwise, set $k = k+1$ and repeat steps 2 to 6.

Step 8: Evaluate the optimal value of n^* taking z_j^* and p_j^* for all $j = 1,2,3, \ldots, N$.

Step 9: Using these results, calculate the optimal values of EAP_V and EAP_S.

4.2. Centralized Model

Centralized decision making gives an efficient management of resource allocation through information sharing and better utilization of resources. Coordination between the supply chain members enhances the supply chain performance by reducing manufacturing cost, transportaion costs, labour costs and lost-sale cost, and improving the product's availability to avoid stock-outs. In the centralized structure, the manufacturer and all the retailers act as a single decision maker without worrying about their own profit levels. They jointly decide the optimal selling prices, number of batch shipments and batch sizes which lead to maximum average expected profit of the entire supply chain. The average expected profit of the supply chain is

$$EAP_s\left(n,z_i,p_i\right) = EAP_v\left(n\right) + \sum_{i=1}^{N} EAP_i\left(z_i,p_i\right)$$

$$= \sum_{i=1}^{N} p_i D_i - \frac{A_v D}{Q} - h_v\left[\frac{D\sum_{i=1}^{N} z_i}{R} + \frac{Q}{2}\left(1-\frac{D}{R}\right) - \sum_{i=1}^{N}\frac{z_i^{\,2}}{2Q_i}\right] - \sum_{i=1}^{N}\frac{h_v D_i \sigma_i}{\sqrt{2\pi}} - \sum_{i=1}^{N}\frac{(A_i + nF)D_i}{Q_i}$$

$$-\sum_{i=1}^{N} nh_i\left[\int_0^{\frac{r_i}{D_i}} \frac{1}{2}\left[\frac{z_i^2}{Q_i} + \frac{2}{n}\left(r_i - D_i l_i\right)\right] f_L(l_i)dl_i + \int_{\frac{r_i}{D_i}}^{\frac{r_i+z_i}{D_i}} \frac{\left(z_i - D_i l_i + r_i\right)^2}{2Q_i} f_L(l_i)dl_i\right]$$

$$-\sum_{i=1}^{N} nc_i\left[\int_{\frac{r_i}{D_i}}^{\frac{r_i+z_i}{D_i}} \frac{\left(D_i l_i - r_i\right)^2}{2Q_i} f_L(l_i)dl_i + \int_{\frac{r_i+z_i}{D_i}}^{\infty}\left[\frac{z_i^2}{2Q_i} + z_i\left(\frac{D_i l_i - z_i - r_i}{Q_i}\right)\right] f_L(l_i)dl_i\right] \tag{12}$$

Proposition 4. The average expected profit function (3) is concave in n for known z_i and p_i if $h_v s^2 < 2\left(A_v D + ms\right)$ and the optimal number of batch shipments is

$$n^* = \sqrt{\frac{R\left(2A_vD - h_vs^2 + 2ms\right)}{h_vs^2\left(R-D\right)}} \text{ where } s = \sum_{i=1}^{N} z_i \text{ and } m = \left(\sum_{i=1}^{N}\frac{A_iD_i}{z_i}\right) \tag{13}$$

Proof. Assuming n as real, not just an integer, we get from (12)

$$\frac{\partial EAP_s}{\partial n} = \frac{A_vD}{n^2s} - \frac{h_vs}{2} + \frac{h_vsD}{2R} - \frac{h_vs}{2n^2} + \frac{m}{n^2} \tag{14}$$

$$\frac{\partial^2 EAP_s}{\partial n^2} = -\frac{2A_vD}{n^3s} + \frac{h_vs}{n^3} - \frac{2m}{n^3} \tag{15}$$

The average expected profit function of the entire supply chain is concave in n if $\frac{\partial^2 EAP_s}{\partial n^2} < 0$, which implies that $h_vs^2 < 2\left(A_vD + ms\right)$.

Now, if the above condition of concavity holds then one can derive the optimal number of batch shipments by using the first order optimality condition. Then, by solving $\frac{\partial EAP_s}{\partial n} = 0$ for n, we get the optimal number of shipments as $n^* = \sqrt{\frac{R\left(2A_vD - h_vs^2 + 2ms\right)}{h_vs^2\left(R-D\right)}}$.

Proposition 5. The average expected system profit function (3) will be concave in z_i for given n and p_i if

$$\frac{2A_vD}{s^3} + \frac{2(A_i + nF)D_i}{z_i^{\,3}} + n\left(h_i + c_i\right)\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{D_i^{\,2}l_i^{\,2}}{z_i^{\,3}} f_L(l_i)dl_i > 0 \tag{16}$$

Proof. Using $r_i = \frac{z_iD_i}{R}$ and differentiating twice equation (12) with respect to z_i, we obtain

$$\frac{\partial EAP_s}{\partial z_i} = \frac{\partial}{\partial z_i}\left[-\frac{A_vD}{Q} - h_v\left[\frac{D\sum_{i=1}^{N} z_i}{R} + \frac{Q}{2}\left(1-\frac{D}{R}\right) - \sum_{i=1}^{N}\frac{z_i^{\,2}}{2Q_i}\right] - \sum_{i=1}^{N}\frac{(A_i + nF)D_i}{Q_i}\right]$$

$$-\frac{\partial}{\partial z_i}\left[\sum_{i=1}^{N} nh_i\left[\int_{0}^{\frac{z_i}{R}}\left(z_i + 2\left(\frac{z_iD_i}{R} - D_il_i\right)\right)f_L(l_i)dl_i + \int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}}\frac{\left(z_i - D_il_i + \frac{z_iD_i}{R}\right)^2}{2Q_i}f_L(l_i)dl_i\right]\right]$$

$$-\frac{\partial}{\partial z_i}\left[\sum_{i=1}^{N} nc_i\left[\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}}\frac{\left(D_il_i - \frac{z_iD_i}{R}\right)^2}{2Q_i}f_L(l_i)dl_i + \int_{\frac{z_i(R+D_i)}{RD_i}}^{\infty}\left[\frac{z_i^2}{2Q_i} + z_i\left(\frac{D_il_i - z_i - \frac{z_iD_i}{R}}{Q_i}\right)\right]f_L(l_i)dl_i\right]\right]$$

$$= \frac{A_v D}{ns^2} - \frac{h_v D}{R} - \frac{nh_v}{2}\left(1-\frac{D}{R}\right) + \frac{h_v}{2n} + \frac{(A_i+nF)D_i}{nz_i^2} - \frac{h_i}{2}\left[\int_0^{\frac{z_i}{R}} \frac{\partial}{\partial z_i}\left[z_i + 2\left(\frac{z_i D_i}{R} - D_i l_i\right)\right] f_L(l_i)dl_i + \left(z_i + 2\left(\frac{z_i D_i}{R} - D_i.\frac{z_i}{R}\right)\right).f_L\left(\frac{z_i}{R}\right)\frac{\partial}{\partial z_i}\left(\frac{z_i}{R}\right)\right]$$

$$-\frac{h_i}{2}\left[\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{\partial}{\partial z_i}\left[\frac{\left(z_i - D_i l_i + \frac{z_i D_i}{R}\right)^2}{z_i}\right] f_L(l_i)dl_i + \frac{\left(z_i - D_i.\frac{z_i(R+D_i)}{RD_i} + \frac{z_i D_i}{R}\right)^2}{z_i} f_L\left(\frac{z_i(R+D_i)}{RD_i}\right)\frac{\partial}{\partial z_i}\left(\frac{z_i(R+D_i)}{RD_i}\right) - \frac{\left(z_i - D_i.\frac{z_i}{R} + \frac{z_i D_i}{R}\right)^2}{z_i} f_L\left(\frac{z_i}{R}\right)\frac{\partial}{\partial z_i}\left(\frac{z_i}{R}\right)\right]$$

$$-\frac{c_i}{2}\left[\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{\partial}{\partial z_i}\left[\frac{\left(D_i l_i - \frac{z_i D_i}{R}\right)^2}{z_i}\right] f_L(l_i)dl_i + \frac{\left(D_i.\frac{z_i(R+D_i)}{RD_i} - \frac{z_i D_i}{R}\right)^2}{z_i}.f_L\left(\frac{z_i(R+D_i)}{RD_i}\right)\frac{\partial}{\partial z_i}\left(\frac{z_i(R+D_i)}{RD_i}\right) - \frac{\left(D_i.\frac{z_i}{R} - \frac{z_i D_i}{R}\right)^2}{z_i}.f_L\left(\frac{z_i}{R}\right)\frac{\partial}{\partial z_i}\left(\frac{z_i}{R}\right)\right]$$

$$-\frac{c_i}{2}\left[\int_{\frac{z_i(R+D_i)}{RD_i}}^{\infty} \frac{\partial}{\partial z_i}\left[z_i + 2\left(D_i l_i - z_i - \frac{z_i D_i}{R}\right)\right] f_L(l_i)dl_i - \left[z_i + 2\left(D_i.\frac{z_i(R+D_i)}{RD_i} - z_i - \frac{z_i D_i}{R}\right)\right] f_L\left(\frac{z_i(R+D_i)}{RD_i}\right)\frac{\partial}{\partial z_i}\left(\frac{z_i(R+D_i)}{RD_i}\right)\right]$$

$$= \frac{A_v D}{ns^2} - \frac{h_v D}{R} - \frac{nh_v}{2}\left(1-\frac{D}{R}\right) + \frac{h_v}{2n} + \frac{(A_i+nF)D_i}{nz_i^2} - \frac{h_i}{2}\int_0^{\frac{z_i}{R}}\left(\frac{R+2D_i}{R}\right) f_L(l_i)dl_i - \frac{h_i}{2}\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}}\left(\frac{z_i^2(R+D_i)^2 - R^2 D_i^2 l_i^2}{z_i^2 R^2}\right) f_L(l_i)dl_i$$

$$+\frac{c_i}{2}\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}}\left(\frac{R^2 D_i^2 l_i^2 - z_i^2 D_i^2}{z_i^2 R^2}\right) f_L(l_i)dl_i + \frac{c_i}{2}\int_{\frac{z_i(R+D_i)}{RD_i}}^{\infty}\left(\frac{R+2D_i}{R}\right) f_L(l_i)dl_i$$

(17)

$$\frac{\partial^2 EAP_s}{\partial z_i^2} = -\frac{2(A_i+nF)D_i}{nz_i^3} - \frac{2A_v D}{ns^3} - \frac{h_i}{2}\left[\int_0^{\frac{z_i}{R}} \frac{\partial}{\partial z_i}\left(\frac{R+2D_i}{R}\right) f_L(l_i)dl_i + \left(\frac{R+2D_i}{R}\right) f_L\left(\frac{z_i}{R}\right)\frac{\partial}{\partial z_i}\left(\frac{z_i}{R}\right)\right]$$

$$-\frac{h_i}{2}\left[\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{\partial}{\partial z_i}\left(\frac{z_i^2(R+D_i)^2 - R^2 D_i^2 l_i^2}{z_i^2 R^2}\right) f_L(l_i)dl_i + \left(\frac{z_i^2(R+D_i)^2 - R^2 D_i^2.\frac{z_i^2(R+D_i)^2}{R^2 D_i^2}}{z_i^2 R^2}\right) f_L\left(\frac{z_i(R+D_i)}{RD_i}\right)\frac{\partial}{\partial z_i}\left(\frac{z_i(R+D_i)}{RD_i}\right) - \left(\frac{z_i^2(R+D_i)^2 - R^2 D_i^2.\frac{z_i^2}{R^2}}{z_i^2 R^2}\right) f_L\left(\frac{z_i}{R}\right)\frac{\partial}{\partial z_i}\left(\frac{z_i}{R}\right)\right]$$

$$+\frac{c_i}{2}\left[\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{\partial}{\partial z_i}\left(\frac{R^2 D_i^2 l_i^2 - z_i^2 D_i^2}{z_i^2 R^2}\right) f_L(l_i)dl_i + \left(\frac{R^2 D_i^2.\frac{z_i^2(R+D_i)^2}{R^2 D_i^2} - z_i^2 D_i^2}{z_i^2 R^2}\right).f_L\left(\frac{z_i(R+D_i)}{RD_i}\right)\frac{\partial}{\partial z_i}\left(\frac{z_i(R+D_i)}{RD_i}\right) - \left(\frac{R^2 D_i^2.\frac{z_i^2}{R^2} - z_i^2 D_i^2}{z_i^2 R^2}\right).f_L\left(\frac{z_i}{R}\right)\frac{\partial}{\partial z_i}\left(\frac{z_i}{R}\right)\right]$$

$$+\frac{c_i}{2}\left[\int_{\frac{z_i(R+D_i)}{RD_i}}^{\infty} \frac{\partial}{\partial z_i}\left(\frac{R+2D_i}{R}\right) f_L(l_i)dl_i - \left(\frac{R+2D_i}{R}\right) f_L\left(\frac{z_i(R+D_i)}{RD_i}\right)\frac{\partial}{\partial z_i}\left(\frac{z_i(R+D_i)}{RD_i}\right)\right]$$

$$= -\frac{2A_v D}{ns^3} - \frac{2(A_i+nF)D_i}{nz_i^3} - \frac{h_i}{2}\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{2D_i^2 l_i^2}{z_i^3} f_L(l_i)dl_i - \frac{c_i}{2}\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{2D_i^2 l_i^2}{z_i^3} f_L(l_i)dl_i \quad (18)$$

Now, the average expected profit of the entire supply chain is concave in z_i if $\frac{\partial^2 EAP_s}{\partial z_i^{\,2}} < 0$, which gives

$$\frac{2A_v D}{s^3} + \frac{2(A_i + nF)D_i}{z_i^{\,3}} + n\left(h_i + c_i\right)\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{D_i^{\,2} l_i^{\,2}}{z_i^{\,3}} f_L(l_i)dl_i > 0.$$

Proposition 6. The average expected system profit function (3) is concave in p_i for given n and z_i if

$$2\beta_i + \left(h_i + c_i\right)\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{\beta_i^{\,2}\left(z_i - Rl_i\right)^2}{z_i R^2} f_L(l_i)dl_i > 0 \tag{19}$$

Proof. Using $r_i = \frac{z_i D_i}{R}$ and differentiating (12) twice with respect to p_i, we have

$$\frac{\partial EAP_s}{\partial p_i} = \frac{\partial}{\partial p_i}\left[\sum_{i=1}^{N} p_i D_i - \frac{A_v D}{Q} - h_v\left[\frac{D\sum_{i=1}^{N} z_i}{R} + \frac{Q}{2}\left(1 - \frac{D}{R}\right) - \sum_{i=1}^{N}\frac{z_i^{\,2}}{2Q_i}\right] - \sum_{i=1}^{N}\frac{h_v D_i \sigma_i}{\sqrt{2\pi}} - \sum_{i=1}^{N}\frac{(A_i + nF)D_i}{Q_i}\right]$$

$$-\frac{\partial}{\partial p_i}\left[\sum_{i=1}^{N} nh_i\left[\int_0^{\frac{z_i}{R}}\left(z_i + 2\left(\frac{z_i D_i}{R} - D_i l_i\right)\right) f_L(l_i)dl_i + \int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{\left(z_i - D_i l_i + \frac{z_i D_i}{R}\right)^2}{2Q_i} f_L(l_i)dl_i\right]\right]$$

$$-\frac{\partial}{\partial p_i}\left[\sum_{i=1}^{N} nc_i\left[\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{\left(D_i l_i - \frac{z_i D_i}{R}\right)^2}{2Q_i} f_L(l_i)dl_i + \int_{\frac{z_i(R+D_i)}{RD_i}}^{\infty}\left[\frac{z_i^2}{2Q_i} + z_i\left(\frac{D_i l_i - z_i - \frac{z_i D_i}{R}}{Q_i}\right)\right] f_L(l_i)dl_i\right]\right]$$

$$= D_i - \beta_i p_i + \frac{(A_i + nF)\beta_i}{Q_i} + \frac{A_v \beta_i}{Q} + \frac{h_v \beta_i s}{R} - \frac{h_v Q \beta_i}{2R} + \frac{h_v \beta_i \sigma_i}{\sqrt{2\pi}} - \frac{h_i}{2}\left[\int_0^{\frac{z_i}{R}} \frac{\partial}{\partial p_i}\left[z_i + 2\left(\frac{z_i D_i}{R} - D_i l_i\right)\right] f_L(l_i)dl_i + \left(z_i + 2\left(\frac{z_i D_i}{R} - D_i.\frac{z_i}{R}\right)\right).f_L\left(\frac{z_i}{R}\right)\frac{\partial}{\partial p_i}\left(\frac{z_i}{R}\right)\right]$$

$$-\frac{h_i}{2}\left[\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{\partial}{\partial p_i}\left[\frac{\left(z_i - D_i l_i + \frac{z_i D_i}{R}\right)^2}{z_i}\right] f_L(l_i)dl_i + \frac{\left(z_i - D_i.\frac{z_i(R+D_i)}{RD_i} + \frac{z_i D_i}{R}\right)^2}{z_i} f_L\left(\frac{z_i(R+D_i)}{RD_i}\right)\frac{\partial}{\partial p_i}\left(\frac{z_i(R+D_i)}{RD_i}\right) - \frac{\left(z_i - D_i.\frac{z_i}{R} + \frac{z_i D_i}{R}\right)^2}{z_i} f_L\left(\frac{z_i}{R}\right)\frac{\partial}{\partial p_i}\left(\frac{z_i}{R}\right)\right]$$

$$-\frac{c_i}{2}\left[\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{\partial}{\partial p_i}\left[\frac{\left(D_i l_i - \frac{z_i D_i}{R}\right)^2}{z_i}\right] f_L(l_i)dl_i + \frac{\left(D_i.\frac{z_i(R+D_i)}{RD_i} - \frac{z_i D_i}{R}\right)^2}{z_i}.f_L\left(\frac{z_i(R+D_i)}{RD_i}\right)\frac{\partial}{\partial p_i}\left(\frac{z_i(R+D_i)}{RD_i}\right) - \frac{\left(D_i.\frac{z_i}{R} - \frac{z_i D_i}{R}\right)^2}{z_i}.f_L\left(\frac{z_i}{R}\right)\frac{\partial}{\partial p_i}\left(\frac{z_i}{R}\right)\right]$$

$$-\frac{c_i}{2}\left[\int_{\frac{z_i(R+D_i)}{RD_i}}^{\infty} \frac{\partial}{\partial p_i}\left[z_i + 2\left(D_i l_i - z_i - \frac{z_i D_i}{R}\right)\right] f_L(l_i)dl_i - \left[z_i + 2\left(D_i.\frac{z_i(R+D_i)}{RD_i} - z_i - \frac{z_i D_i}{R}\right)\right] f_L\left(\frac{z_i(R+D_i)}{RD_i}\right)\frac{\partial}{\partial p_i}\left(\frac{z_i(R+D_i)}{RD_i}\right)\right]$$

$$= D_i - \beta_i p_i + \frac{A_v \beta_i}{Q} + \frac{h_v \beta_i s}{R} - \frac{h_v Q \beta_i}{2R} + \frac{h_v \beta_i \sigma_i}{\sqrt{2\pi}} + \frac{(A_i + nF)\beta_i}{Q_i} + h_i \int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{\beta_i\left(z_i - Rl_i\right)\left[z_i R + D_i\left(z_i - Rl_i\right)\right]}{z_i R^2} f_L(l_i)dl_i$$

$$+ h_i \int_0^{\frac{z_i}{R}} \frac{\beta_i\left(z_i - Rl_i\right)}{R} f_L(l_i)dl_i + c_i \int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{D_i\beta_i\left(z_i - Rl_i\right)^2}{z_i R^2} f_L(l_i)dl_i - c_i \int_{\frac{z_i(R+D_i)}{RD_i}}^{\infty} \frac{\beta_i\left(z_i - Rl_i\right)}{R} f_L(l_i)dl_i \tag{20}$$

$$\frac{\partial^2 EAP_s}{\partial p_i^2} = h_i \left[\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{\partial}{\partial p_i}\left(\frac{\beta_i\left(z_i - Rl_i\right)\left[z_i R + D_i\left(z_i - Rl_i\right)\right]}{z_i R^2} \right) f_L(l_i)dl_i + \frac{\beta_i\left(z_i - R.\frac{z_i(R+D_i)}{RD_i}\right)\left[z_i R + D_i\left(z_i - R.\frac{z_i(R+D_i)}{RD_i}\right)\right]}{z_i R^2} f_L\left(\frac{z_i(R+D_i)}{RD_i}\right)\frac{\partial}{\partial p_i}\left(\frac{z_i(R+D_i)}{RD_i}\right) \right]$$

$$+ c_i \left[\int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{\partial}{\partial p_i}\left(\frac{D_i\beta_i\left(z_i - Rl_i\right)^2}{z_i R^2} \right) f_L(l_i)dl_i + \left(\frac{D_i\beta_i\left(z_i - R.\frac{z_i(R+D_i)}{RD_i}\right)^2}{z_i R^2} \right) f_L\left(\frac{z_i(R+D_i)}{RD_i}\right)\frac{\partial}{\partial p_i}\left(\frac{z_i(R+D_i)}{RD_i}\right) \right]$$

$$- c_i \left[\int_{\frac{z_i(R+D_i)}{RD_i}}^{\infty} \frac{\partial}{\partial p_i}\left(\frac{\beta_i\left(z_i - Rl_i\right)}{R} \right) f_L(l_i)dl_{ii} - \left(\frac{\beta_i\left(z_i - R.\frac{z_i(R+D_i)}{RD_i}\right)}{R} \right) f_L\left(\frac{z_i(R+D_i)}{RD_i}\right)\frac{\partial}{\partial p_i}\left(\frac{z_i(R+D_i)}{RD_i}\right) \right] - 2\beta_i$$

$$= -2\beta_i - \left(h_i + c_i\right) \int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{\beta_i^2\left(z_i - Rl_i\right)^2}{z_i R^2} f_L(l_i)dl_i \tag{21}$$

The average expected profit of entire system is concave in p_i if $\frac{\partial^2 EAP_s}{\partial p_i^2} < 0$ i.e. if

$$2\beta_i + \left(h_i + c_i\right) \int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{\beta_i^2\left(z_i - Rl_i\right)^2}{z_i R^2} f_L(l_i)dl_i > 0 \cdot$$

Proposition 7. The average expected profit function (12) is jointly concave with respect to n, z_i and p_i if

(i) $h_v s^2 < 2\left(A_v D + ms\right)$

(ii) $2\left(h_i + c_i\right)X_4 X_5 > X_4^2 + 2\beta_i X_2 + \left(h_i + c_i\right)\left(X_2 X_3 + \left(h_i + c_i\right)X_5^2\right)$

(iii) $\left(h_i + c_i\right)X_8 + 2\beta_i X_1 X_2 + X_1 X_4^2 + X_2 X_7^2 > \left(h_i + c_i\right)X_9 + 2X_4 X_6 X_7 + 2\beta_i X_6^2$ where

$$m = \sum_{i=1}^{N} \frac{A_i D_i}{z_i} \tag{22}$$

$$s = \sum_{i=1}^{N} z_i \tag{23}$$

$$X_1 = -\frac{2A_v D}{n^3 s} + \frac{h_v s}{n^3} - \frac{2m}{n^3} \tag{24}$$

$$X_2 = -\frac{2A_v D}{ns^3} - \frac{2(A_i + nF)D_i}{nz_i^3} - \left(h_i + c_i\right) \int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{D_i^2 l_i^2}{z_i^3} f_L(l_i) dl_i \tag{25}$$

$$X_3 = \int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{\beta_i^2 \left(z_i - Rl_i\right)^2}{z_i R^2} f_L(l_i) dl_i \tag{26}$$

$$X_4 = -\frac{(A_i + nF)\beta_i}{nz_i^2} - \frac{A_v \beta_i}{ns^2} + \frac{h_v \beta_i}{R} - \frac{nh_v \beta_i}{2R} + h_i \int_0^{\frac{z_i(R+D_i)}{RD_i}} \frac{\beta_i}{R} f_L(l_i) dl_i - c_i \int_{\frac{z_i(R+D_i)}{RD_i}}^{\infty} \frac{\beta_i}{R} f_L(l_i) dl_i \tag{27}$$

$$X_5 = \int_{\frac{z_i}{R}}^{\frac{z_i(R+D_i)}{RD_i}} \frac{\beta_i D_i \left(R^2 l_i^2 - z_i^2\right)}{z_i^2 R^2} f_L(l_i) dl_i \tag{28}$$

$$X_6 = -\frac{A_v D}{n^2 s^2} - \frac{h_v}{2}\left(1 - \frac{D}{R}\right) - \frac{h_v}{2n^2} - \frac{A_i D_i}{n^2 z_i^2} \tag{29}$$

$$X_7 = -\frac{A_v \beta_i}{n^2 s} - \frac{h_v \beta_i s}{2R} - \frac{A_i \beta_i}{n^2 z_i} \tag{30}$$

$$X_8 = \left(X_1 X_2 X_3 + 2X_5 X_6 X_7 + \left(h_i + c_i\right) X_1 X_5^2\right) \tag{31}$$

$$X_9 = \left(2X_1 X_4 X_5 + X_3 X_6^2\right) \tag{32}$$

Proof. From equation (12), we have

$$\frac{\partial^2 EAP_s}{\partial n^2} = X_1 \tag{33}$$

$$\frac{\partial^2 EAP_s}{\partial z_i^2} = X_2 \tag{34}$$

$$\frac{\partial^2 EAP_s}{\partial p_i^2} = -2\beta_i - \left(h_i + c_i\right) X_3 \tag{35}$$

$$\frac{\partial^2 EAP_s}{\partial z_i \partial p_i} = \frac{\partial^2 EAP_s}{\partial p_i \partial z_i} = X_4 - \left(h_i + c_i\right) X_5 \tag{36}$$

$$\frac{\partial^2 EAP_s}{\partial n \partial z_i} = \frac{\partial^2 EAP_s}{\partial z_i \partial n} = X_6 \tag{37}$$

$$\frac{\partial^2 EAP_s}{\partial n \partial p_i} = \frac{\partial^2 EAP_s}{\partial p_i \partial n} = X_7 \tag{38}$$

The Hessian matrix associated with the average expected profit function $EAP_s(n, z_i, p_i)$ is obtained as

$$H=\begin{pmatrix} \frac{\partial^2 EAP_s}{\partial n^2} & \frac{\partial^2 EAP_s}{\partial n \partial z_i} & \frac{\partial^2 EAP_s}{\partial n \partial p_i} \\ \frac{\partial^2 EAP_s}{\partial z_i \partial n} & \frac{\partial^2 EAP_s}{\partial {z_i}^2} & \frac{\partial^2 EAP_s}{\partial z_i \partial p_i} \\ \frac{\partial^2 EAP_s}{\partial p_i \partial n} & \frac{\partial^2 EAP_s}{\partial p_i \partial z_i} & \frac{\partial^2 EAP_s}{\partial {p_i}^2} \end{pmatrix}=\begin{pmatrix} X_1 & X_6 & X_7 \\ X_6 & X_2 & X_4-(h_i+c_i)X_5 \\ X_7 & X_4-(h_i+c_i)X_5 & -2\beta_i-(h_i+c_i)X_3 \end{pmatrix} \tag{39}$$

Now, $\frac{\partial^2 EAP_s}{\partial n^2}=X_1<0$ if $h_v s^2<2(A_v D+ms)$. (40)

The second order minor $=|H_2|=\begin{vmatrix} X_2 & X_4-(h_i+c_i)X_5 \\ X_4-(h_i+c_i)X_5 & -2\beta_i-(h_i+c_i)X_3 \end{vmatrix}>0$ if

$$2(h_i+c_i)X_4X_5>{X_4}^2+2\beta_iX_2+(h_i+c_i)\left(X_2X_3+(h_i+c_i){X_5}^2\right) \tag{41}$$

and $|H|<0$ only if

$$(h_i+c_i)X_8+2\beta_iX_1X_2+X_1{X_4}^2+X_2{X_7}^2>(h_i+c_i)X_9+2X_4X_6X_7+2\beta_i{X_6}^2 \tag{42}$$

Therefore, the Hessian matrix H is negative definite if the above conditions (40), (41) and (42) are satisfied. It ensures the concavity of the average expected profit of the entire supply chain with respect to n, z_i and p_i and existence of unique optimal solution.

5. Numerical Analysis

To analyze the proposed model numerically, an example is taken with the following set of parameter- values:

$R=2500$ units / year;	$A_v=\$500$ / set up;	$c_1=\$6$ / unit;
$c_2=\$6$ / unit;	$a_1=1000$ units / year;	$a_2=1000$ units / year;
$w=\$80$ / unit;	$h_1=\$5$ per unit / year;	$h_2=\$4.8$ / unit / year;
$F=\$10$ / shipment;	$h_v=\$3.5$ per unit / year;	$A_1=\$50$ / order;
$A_2=\$45$ / order;	$\sigma_1=0.12$;	$\sigma_2=0.13$;
$\beta_1=3.5$;	$\beta_2=4.5$;	$N=2$;

The p. d. f. of the lead time of i-th retailer is assumed as $f_L(l_i)=\frac{1}{\sqrt{2\pi}\sigma_i}e^{-\frac{1}{2{\sigma_i}^2}\left(l_i-\frac{r_i}{D_i}\right)^2}$.

Here we have studied a system consisting of a single manufacturer and two retailers. The concavity of the profit function of the whole supply chain system with respect to its decision variables is observed for the chosen parameter-values. Figure 3 exhibits the concavity of the average expected profit function of the entire supply chain with respect to n. The nature of the profit functions of both the retailers has been found to be concave for different values of (z_1, p_1), and (z_2, p_2) in all situations. One instance for each retailer is reflected in Figures 4 and 5.

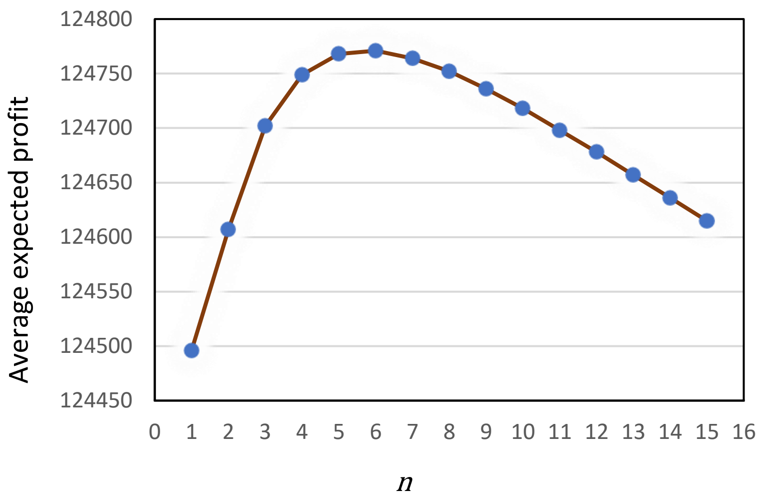


Figure 3. Concavity of the average expected profit of the supply chain w.r.t. n

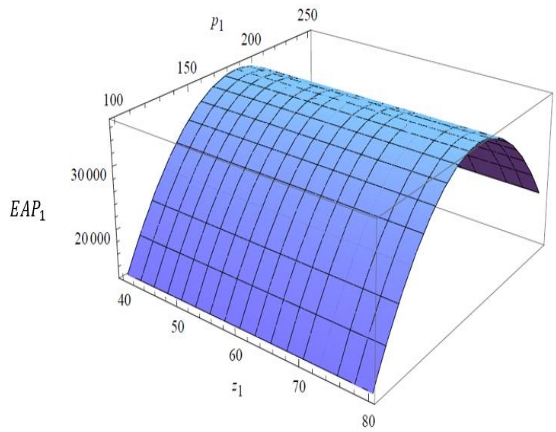


Figure 4. Concavity of the average expected profit function of the first retailer

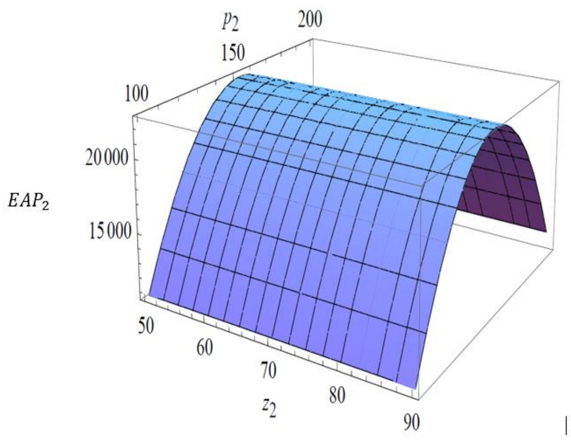


Figure 5. Concavity of the average expected profit function of the second retailer

Table 2. Optimal results for different models

Models	n^*	z_1^*	z_2^*	p_1^*	p_2^*	EAP_1^*	EAP_2^*	EAP_V^*	EAP_S^*
Decentralized	4	69.29	74.42	183.01	151.26	36821	22590	52805	112215
Centralized	6	60.07	63.49	143.33	111.59	-	-	-	124771

Using the prescribed solution algorithm, we find the optimal batch sizes as well as the order quantities of the retailers, selling price of the product and the number of batch shipments in the decentralized model. Table 2 shows the optimal results of both the decentralized and centralized models. It can be observed from Table 2 that the profit in the centralized model is more than the profit in the decentralized model. In our proposed model, we assume that, the customer demand is linearly dependent on its retail price. In the centralized decision making scenario, the manufacturer and all the retailers act as a single business manager and jointly make their optimal decisions in order to achieve highest system profit. So, the retailers can provide the product to the customers at a cheaper price than that of the decentralized case. That's why lower priced product increases the consumer demand significantly. Then the retailers order more quantity from the manufacturer. As a result, integration between the manufacturer and the retailers increases the total system profit significantly. We observe that the optimal ordering quantity of the first retailer is increased by 83.26 units and for the second retailer, it is increased by 83.26 units in the centralized model. Moreover, the retail prices of the product also decrease for both the retailers. This results in an increase of profit for the entire supply chain.

6. Sensitivity Analysis

The impacts of several important parameters on the expected average profit of the supply chain are analyzed in this section. We change one parameter-value at a time, keeping other parameter-values fixed. The results of sensitivity analysis are presented in Table 3. From Table 3 we have the following observations:

(i) The expected average profit of the entire supply chain is highly sensitive with respect to the parameters β_1 and β_2. It can be noticed that, both the retailers reduce their selling prices of the product when the values of β_1 and β_2 increase. As higher values of β_1 and β_2 indicate that the customers prefer to purchase cheaper products, so in order to keep the consumer demand intact, the retailers reduce their selling prices. Also, the ordered quantities of both the retailers decrease as β_1 and β_2 increase. Consequently, the average expected profits of all individual supply chain players as well as the entire supply chain decrease gradually as β_1 and β_2 increase.

(ii) The impacts of the basic market demands a_1 and a_2 on the average expected profit of the supply chain can be realised from the optimal results given in Table 3. Whenever the customer demand increases, the retailers place orders for more items from the manufacturer. Also, with the increment of the basic market demand, the retailers get the opportunity to raise the retail prices and earn more revenue. This leads the supply chain system to a better profit level.

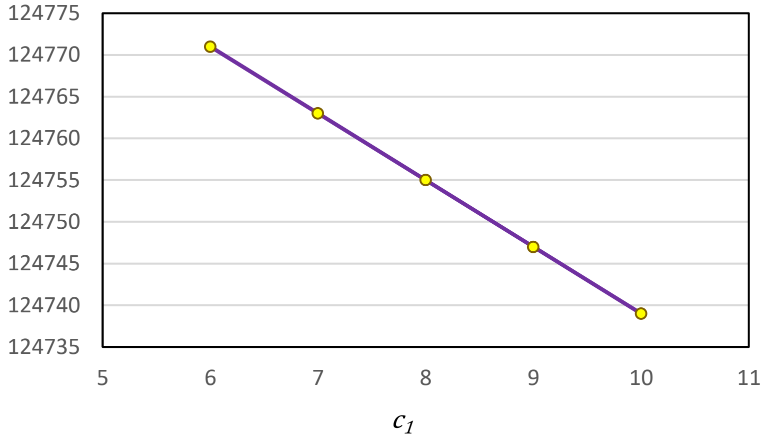


Figure 6. Average expected profit of the supply chain vs. c_1

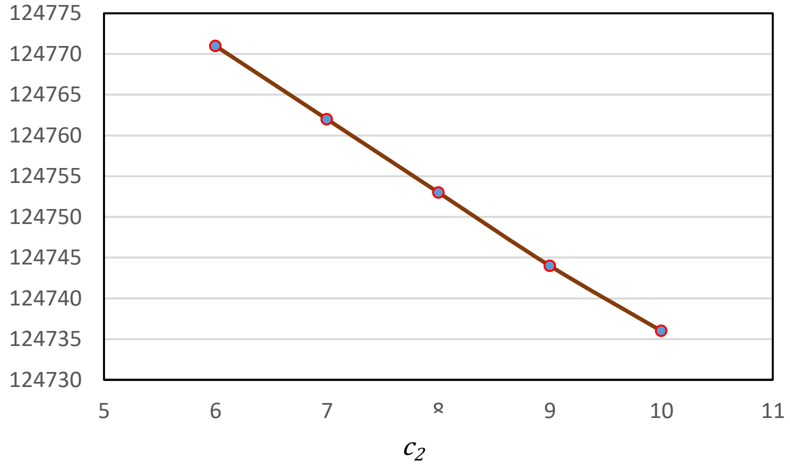


Figure 7. Average expected profit of the supply chain vs. c_2

(iii) From Figure 6 and 7, it can be seen that the average expected profit decreases while shortage costs c_1 and c_2 increase. It is quite natural that, when the shortage cost increases, the retailers increase their order quantities to avoid shortage. But the retail prices remain almost the same for all values of shortage cost.

Table 3. Optimal results for different values of β_1, β_2, a_1 and a_2

Parameter	Value	n	z_1	z_2	p_1	p_2	EAP_S
	3.1	6	59.4274	62.3693	161.782	111.597	133999
	3.3	6	60.0768	63.4912	151.984	111.585	129100
β_1	3.5	6	60.0696	63.4904	143.326	111.585	124771
	3.7	6	60.0624	63.4896	135.604	111.585	120910
	3.9	6	60.0552	63.4889	128.674	111.585	117445
	4.1	6	59.4274	62.3693	161.782	111.597	133999
	4.3	6	60.0768	63.4912	151.984	111.585	129100
β_2	4.5	6	60.0696	63.4904	143.326	111.585	124771
	4.7	6	60.0624	63.4896	135.604	111.585	120910
	4.9	6	60.0552	63.4889	128.674	111.585	117445
	800	6	52.1983	62.7725	114.803	111.625	99154
	900	6	56.1819	63.1056	129.063	111.604	111247
a_1	1000	6	60.0696	63.4904	143.326	111.585	124771
	1100	6	63.8784	63.9203	157.59	111.566	139725
	1200	6	67.6216	63.3906	171.856	111.548	156110
	800	6	59.6716	55.0633	143.366	89.411	104870
	900	6	59.8473	59.3196	143.345	100.497	114264
a_2	1000	6	60.0696	63.4904	143.326	111.585	124771
	1100	6	60.3313	67.5938	143.307	122.675	136391
	1200	6	67.6272	71.6443	143.289	133.766	149125

(iv) From Figure 8, we notice that the transportation cost affects the optimal number shipments significantly. As the transportation cost increases, the optimal number of batch shipments decreases. The average expected profit of the supply chain also decreases with increased transportation cost.

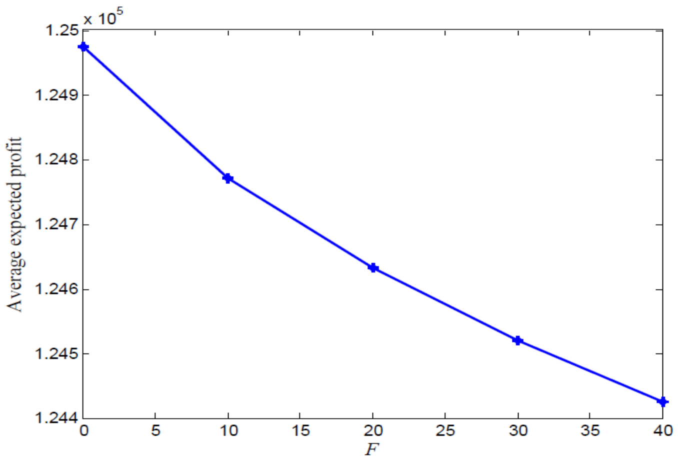


Figure 8. Average expected profit of the supply chain vs. F

(v) The standard deviations of the lead times of both the retailers do not affect the profit of the supply chain considerably. When the standard deviations σ_1 and σ_2 increase approximately by 36% and 33%, respectively, the average system profit decreases approximately by 4%.

7. Conclusion

In this paper, we have considered a two-level supply chain model consisting of a single manufacturer and multiple retailers under price-sensitive customer demand and stochastic lead time. The manufacturer produces a single product at a constant production rate and transfers it to the retailers in some equal batch shipments. Due to the stochastic nature of lead time the batches may reach early or late to the retailers. Depending on the length of lead time duration, three possible cases take into consideration at the retailers' end. Both the decentralized and centralized models are formulated. A solution algorithm is suggested to find the optimal solution of the decentralized model with a Stackelberg gaming structure. It is found that cooperation between the manufacturer and the retailers becomes more profitable for the entire supply chain whenever the consumer demand is price dependent and lead time of delivering the ordered quantity follows a normal distribution. By collaborating with each other, the supply chain players can sell the product at a cheaper price to the end customers and enhance the market demand. This leads to a significant increment of the profitability of the whole supply chain.

Our proposed model is well-connected to the real world business environment. The present study would be helpful to companies to make the efficient management of supply chains in stochastic environments. The proposed model provides effective pricing strategies in a highly price-sensitive market. Companies can adopt integrated decision making to achieve the highest system profit and make their products more acceptable to the end customer at a lower price. Also, they can achieve higher customer service levels and survive in today's competitive market by efficiently managing the stochastic nature of lead time.

In our proposed model, we have considered that one manufacturer is trading with multiple retailers for a single product. Our model can be further extended considering multiple items. One can also implement any suitable coordination scheme between the manufacturer and all the retailers to improve their profits in the decentralized scenario. Furthermore, our proposed model can be enriched by considering more realistic assumptions, such as combined equal and unequal sized batch shipments, quality dependent demand and multiple manufacturers trading with multiple retailers.

References

Alfares H. K. and Ghaithan A. M. (2016). Inventory and pricing model with price-dependent demand, time-varying holding cost, and quantity discounts. *Computers & Industrial Engineering*, Vol. 94, pp. 170-177.

Banerjee A. (1986). A joint economic-lot-size model for purchaser and vendor. *Decision sciences*, Vol. 17(3), pp. 292-311.

Banerjee A. (2005). Concurrent pricing and lot sizing for make-to-order contract production. *International Journal of Production Economics*, Vol. 93, pp. 189-195.

Ben-Daya M. and Hariga M. (2004). Integrated single vendor single buyer model with stochastic demand and variable lead time. *International Journal of Production Economics*, Vol. 92(1), pp. 75-80.

Biswas S. and Giri B. C. (2019). A Single-vendor Single-buyer Integrated Model for Deteriorating Items with Partial Backlogging and Price-dependent Market Demand. *International Journal of Applied Engineering Research*, Vol. 14(1), pp. 254-261.

Das S. C., Zidan A. M., Manna A. K., Shaikh A. A. and Bhunia A. K. (2020). An application of preservation technology in inventory control system with price dependent demand and partial backlogging. *Alexandria Engineering Journal*, Vol. 59(3), pp. 1359-1369.

Giri B.C. and Roy B. (2015). A single-manufacturer multi-buyer supply chain inventory model with controllable lead time and price-sensitive demand. *Journal of Industrial and Production Engineering*, Vol. 32(8), pp. 516-527.

Giri B. C., Bhattacharjee R. and Maiti T. (2018). Optimal payment time in a two-echelon supply chain with price-dependent demand under trade credit financing. *International Journal of Systems Science: Operations & Logistics*, Vol. 5(4), pp. 374-392.

Giri B. C. and Masanta M. (2018). Developing a closed-loop supply chain model with price and quality dependent demand and learning in production in a stochastic environment. *International Journal of Systems Science: Operations & Logistics*, pp. 1-17.

Giri B. C. and Masanta M. (2019). Optimal production policy for a closed-loop supply chain with stochastic lead time and learning in production. *Scientia Iranica*, Vol. 26(5), pp. 2936-2951.

Glock C. H. (2012). The joint economic lot size problem: A review. *International Journal of Production Economics*, Vol. 135(2), pp. 671-686.

Glock C. H. (2012). Lead time reduction strategies in a single-vendor-single-buyer integrated inventory model with lot size-dependent lead times and stochastic demand. *International Journal of Production Economics,* Vol. 136(1), pp. 37-44.

Goyal S. K. (1976). An integrated inventory model for a single supplier-single customer problem. *The International Journal of Production Research*, Vol. 15(1), pp. 107-111.

Goyal S. K., and Nebebe F. (2000). Determination of economic production-shipment policy for a single-vendor-single-buyer system. *European Journal of Operational Research*, Vol. 121(1), pp. 175-178.

Ho C. H. (2009). A minimax distribution free procedure for an integrated inventory model with defective goods and stochastic lead time demand. *International Journal of Information and Management Sciences*, Vol. 20(1), pp. 161-171.

Hoque M. A. (2013). A manufacturer-buyer integrated inventory model with stochastic lead times for delivering equal-and/or unequal-sized batches of a lot. *Computers & Operations Research*, Vol. 40(11), pp. 2740-2751.

Khalilpourazari S., Pasandideh S. H. R. and Niaki S. T. A. (2019a). Optimizing a multi-item economic order quantity problem with imperfect items, inspection errors, and backorders. *Soft Computing*, Vol.23(22), pp. 11671-11698.

Khalilpourazari S., Mirzazadeh A., Weber G. W. and Pasandideh S. H. R. (2019b). A robust fuzzy approach for constrained multi-product economic production quantity with imperfect items and rework process. *Optimization*. https://doi.org/10.1080/02331934.2019.1630625

Khalilpourazari S., Teimoori S., Mirzazadeh A., Pasandideh S. H. R. and Ghanbar Tehrani N. (2019c). Robust Fuzzy chance constraint programming for multi-item EOQ model with random disruption and partial backordering under uncertainty. *Journal of Industrial and Production Engineering*, Vol.36(5), pp. 276-285.

Khalilpourazari S., Soltanzadeh S., Weber G. W. and Roy, S. K. (2020). Designing an efficient blood supply chain network in crisis: neural learning, optimization and case study. *Annals of Operations Research*, Vol.289(1), pp. 123-152.

Khanra S., Sana S. S. and Chaudhuri K. (2010). An EOQ model for perishable item with stock and price dependent demand rate. *International Journal of Mathematics in Operational Research*, Vol. 2(3), pp. 320-335.

Jha J. K. and Shanker K. (2013). Single-vendor multi-buyer integrated production-inventory model with controllable lead time and service level constraints. *Applied Mathematical Modelling*, Vol. 37(4), pp. 1753-1767.

Lee W. C., Wu J. W. and Lei C. L. (2007). Computational algorithmic procedure for optimal inventory policy involving ordering cost reduction and back-order discounts when lead time demand is controllable. *Applied mathematics and computation*, Vol. 189(1), pp. 186-200.

Li Y., Xu X. and Ye F. (2011). Supply chain coordination model with controllable lead time and service level constraint. *Computers & Industrial Engineering*, Vol. 61(3), pp. 858-864.

Lin H. J. (2016). Investing in lead-time variability reduction in a collaborative vendor-buyer supply chain model with stochastic lead time. *Computers & Operations Research*, Vol. 72, pp. 43-49.

Mandal P. and Giri B. C. (2015). A single-vendor multi-buyer integrated model with controllable lead time and quality improvement through reduction in defective items. *International Journal of Systems Science: Operations & Logistics*, Vol. 2(1), pp. 1-14.

Maragatham M. and Lakshmidevi P. K. (2014). A fuzzy inventory model for deteriorating items with price dependent demand. *Intern. J. Fuzzy Mathematical Archive*, Vol. 5(1), pp. 39-47.

Maragatham M. and Palani R. (2017). An Inventory Model for Deteriorating Items with Lead Time price Dependent Demand and Shortages. *Advances in Computational Sciences and Technology,* Vol. 10(6), pp. 1839-1847.

Mashud A. H. M., Wee H. M. and Huang C. V. (2019). Preservation technology investment, trade credit and partial backordering model for a non-instantaneous deteriorating inventory. *RAIRO-Operations Research.* https://doi.org/10.1051/ro/2019095

Mashud A. H. M., Wee H. M., Sarkar B. and Li Y. H. C. (2020). A sustainable inventory system with the advanced payment policy and trade-credit strategy for a two-warehouse inventory system. *Kybernetes*. https://doi.org/10.1108/K-01-2020-0052

Ouyang L. Y., Chuang B. R. and Lin Y. J. (2007). The inter-dependent reductions of lead time and ordering cost in periodic review inventory model with backorder price discount. *International Journal of Information and Management Sciences*, Vol. 18(3), pp. 195-208.

Pandey A., Masin M. and Prabhu V. (2007). Adaptive logistic controller for integrated design of distributed supply chains. *Journal of Manufacturing Systems*, Vol. 26(2), pp. 108-115.

Pervin, M., Roy S. K. and Weber G. W. (2019). Multi-item deteriorating two-echelon inventory model with price-and stock-dependent demand: A trade-credit policy. *Journal of Industrial and Management Optimization*, Vol.15(3), pp. 1345-1373.

Pervin M., Roy S. K. and Weber G. W. (2020). Deteriorating inventory with preservation technology under price-and stock-sensitive demand. *Journal of Industrial and Management Optimization*, Vol. 16(4), pp. 1585-1612.

Poorbagheri T. and Niaki S. T. A. (2014). Vendor Managed inventory of a single-vendor multiple-retailer single-warehouse Supply Chain under stochastic Demands. *International Journal of Supply and Operations Management*, Vol. 1(3), pp. 297-313.

Rastogi M., Singh S. R. and Kushwah P. (2018). An inventory model for non-instantaneous deteriorating products having price sensitive demand and partial backlogging of occurring shortages. *Int. J. Oper. Quant. Manag*, Vol. 24(1), pp. 59-73.

Ray S. and Jewkes E. M. (2004). Customer lead time management when both demand and price are lead time sensitive. *European Journal of operational research*, Vol. 153(3), pp. 769-781.

Roy S. K., Pervin M. and Weber G. W. (2018). A two-warehouse probabilistic model with price discount on backorders under two levels of trade-credit policy. *Journal of Industrial and Management Optimization*, Vol. 16(2), pp. 553-578.

Sajadieh M. S., Jokar M. R. A. and Modarres M. (2009). Developing a coordinated vendor-buyer model in two-stage supply chains with stochastic lead-times. *Computers & Operations Research*, Vol. 36(8), pp. 2484-2489.

Sarkar S., Tiwari S., Wee H. M. and Giri B. C. Channel coordination with price discount mechanism under price-sensitive market demand. *International Transactions in Operational Research,* DOI: 10.1111/itor.12678.

Sohrabi M. S., Fattahi P., Kheirkhah A. S. and Esmaeilian G. (2016). Supplier Selection in Three Echelon Supply Chain & Vendor Managed Inventory Model Under Price Dependent Demand Condition. *International Journal of Supply and Operations Management*, Vol. 2(4), pp. 1079-1101.

Taleizadeh A. A., Niaki S. T. A. and Makui A. (2012). Multiproduct multiple-buyer single-vendor supply chain problem with stochastic demand, variable lead-time, and multi-chance constraint. *Expert Systems with Applications*, Vol. 39(5), pp. 5338-5348.

Uthayakumar R. and Kumar M. G. (2019). Single-vendor multi-buyer integrated inventory system for multi-item. *International Journal of Mathematics in Operational Research*, Vol. 14(3), pp. 359-376.

Vijayashree M. and Uthayakumar R. (2015). Integrated inventory model with controllable lead time involving investment for quality improvement in supply chain system. *International Journal of Supply and Operations Management*, Vol. 2(1), pp. 617-639.

Vijayashree M. and Uthayakumar R. (2017). Lead time reduction in an integrated inventory model for non-defective items under a supply chain system. *Computer Game Development and Education: An International Journal*, Vol. 1, pp. 39-58.

Wangsa I. D. and Wee H. M. (2019). A vendor-buyer inventory model for defective items with errors in inspection, stochastic lead time and freight cost. *INFOR: Information Systems and Operational Research*, pp. 1-26.

Whitin T. M. (1955). Inventory control and price theory. *Management science*, Vol. 2(1), pp. 61-68.

Yang C. T., Ouyang L. Y. and Wu H. H. (2009). Retailer's optimal pricing and ordering policies for non-instantaneous deteriorating items with price-dependent demand and partial backlogging. *Mathematical Problems in Engineering*, https://doi.org/10.1155/2009/198305

Yang P. C., Chung S. L., Wee H. M., Zahara E. and Peng C. Y. (2013). Collaboration for a closed-loop deteriorating inventory supply chain with multi-retailer and price-sensitive demand. *International Journal of Production Economics*, Vol. 143(2), pp. 557-566.