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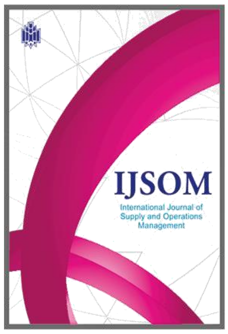

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An Economic Order Quantity model for Decaying Products with the Frequency of Advertisement, Selling Price and Continuous Time Dependent Demand under Partially Backlogged Shortage

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Abstract
In this paper, an inventory model is proposed with consideration of price, frequency of advertisement and continuous time dependent demand is considered. The shortages is partially backlogged. Deterioration is a very important factors for the products, so to manage the inventories in more precise way the retailer always facing some challenges regarding deterioration. In this model, the instantaneous deterioration of the products is considered to give some managerial insights to the retailer how can he manage his inventories in profitable way. Some theoretical derivations together with some numerical example is presented in the paper. To show the convexity of the cost function some graphs are presented in the proposed model. Finally, to check the validity range some of the important parameters of the model, a sensitivity analysis has been carried out.

Keywords: Inventory; Constant deterioration; Partially backlogged shortages; Price dependent demand; Continuous time and frequency of advertisement.

1. Introduction

Traditionally, we define deterioration as decay, change, damage, spoilage or obsolescence of products with consequences in decreasing usefulness from its original purpose. There are many products (e.g., vegetables, fruit, milk, and others) usually deteriorate with time, those are need to handle by retailer with more challenges. Nowadays, according to the existing literature many studies have been carried out on different types of inventory models by considering the product as a deteriorating item. But in the real life situation, deterioration is a critical factor in inventory analysis. However, it has an implication to the inventory system because decayed products are not useable. Economic order quantity model which has a constant rate of deterioration and constant rate of demand over a finite planning horizon was first introduced by Ghare and Schrader (1963). Then Covert and Philip (1973) extended Ghare and Schrader’s constant deterioration rate to a two-parameter weibull distribution deterioration rate. Likewise, there is more research on constant deterioration such as those by Goyal and Giri (2001), Li et al.(2010), Balkhi (2011), Musa and Sani (2012), Taleizadeh et al. (2013), and Mashud et al. (2018).In the present competitive market, the effect of marketing policies and conditions such as the price variations and the advertisement of an item, change its demand pattern amongst the customers. Newspapers, Magazines, radio, and TV along with sales representatives have a motivational effect on the people to buy more. Also, the selling price of an item is one of the decisive factors in selecting an item for use. It is commonly observed that lower selling price causes an increase in demand, whereas higher selling price has the reverse effect. Hence, it can be concluded that the demand of an item is a function of displayed inventory in a show-room, selling price of an item and the advertisement expenditures. There is scanty research on the effects of price variations and the advertisement on the demand rate of items. Kotler (1971) incorporated marketing policies into inventory decisions and discussed the relationship between economic order quantity and pricing decision. Ladany and Sternleib (1974) studied the effect of price variation on selling and consequently on EOQ. However, they did not consider the effect of advertisement. Subramanyam and Kumaraswamy (1981), Urban (1992), Goyal and Gunasekaran (1995), Abad (1996) and Luo (1998), Pal et al. (2007), and Bhunia and Shaikh (2011) developed inventory models incorporating the effects of price variations and advertisement on demand rate of an item.

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Time is an essential factor in inventory management. In fact, there are so many physical goods, which deteriorate over time due to different factors (dryness, damage, spoilage, vaporization, etc.) during the stock-in period. Another vital aspect is the demand which occasionally depends on time. When a new product is introduced to customer then customer has very few interest to this product because of customer always suffer from dilemma for new one. But after some time its demand increases proportionally with the quality of the product. Thus, in the study of inventory problems, this effect cannot be overlooked. Numerous inventory items (for example, electronic goods, and fashionable clothes.) experience variations in the demand rate. Several products experience a period of increasing demand throughout the growth phase of their product life cycle. Alternatively, the demand of some products may fall due to more eye-catching products determining customers' choice. Furthermore, the age of the inventory has an adverse impression on the demand due to loss of consumer self-confidence on the class of such products and physical damage of ingredients. So the researcher improving inventory model of deteriorating items by considering time varying demand in order to regain the confidence of consumers.

In emerging inventory models, two kinds of time varying demands have been considered: (i) discrete-time and (ii) continuous-time. Most of the continuous-time inventory models are developed bearing in mind either linearly increasing/decreasing demand or exponentially increasing/decreasing demand patterns. The thought of exponentially decreasing demand for an inventory model was first projected by Hollier and Mak (1983), who assimilated optimal replenishment policies under both constant and variable replenishment intervals. Hariga and Benkherouf (1994) extended Hollier and Mak's model (1983) by taking into account both exponentially growing and diminishing markets. Wee (1995a) developed a deterministic lot size model for deteriorating items where demand decays exponentially over a fixed time horizon. Thereafter, a deteriorating inventory model is presented by wee et al. (1995b) where demand decreases exponentially with time and cost of items.

Giri et al. (2003) introduced an inventory model with shortage, ramp-type demand rate and time-dependent deterioration rate. Taking in consideration demand as time-dependent, Manna and Chaudhuri (2006) have presented an EOQ model for deteriorating items. An inventory model with weibull deterioration rate, general ramp type demand rate and partial backlogging was considered by Skouri et al. (2009). To show the deterioration effect with time an EOQ model with partial backlogging rate and time varying deterioration is developed by Sana (2010b).. Sarkar (2012a) introduced an EOQ model for finite replenishment rate and concluded that the demand and deterioration rate were both dependent on time. Sett et al. (2012) developed two warehouse inventory models with time-dependent deterioration and quadratic demand. Pervin et al. (2017) worked on an EOQ model with variable holding cost and offering time period allowing delay in payments.

Table 1. Some related research on selling price, time and frequency of advertisement dependent demand.

Author	**Demand function Dependent on**			**Instantaneous deterioration**	**Partial Backlogging**
	selling price	Time	Advertisement frequency		
Hariga (1995)	No	Yes	No	Yes	No
Das et. all (1999)	Yes	No	No	No	No
Jaggi et. all (2012)	Yes	No	No	Yes	Yes
Vijayashree and (2015)	No	No	No	Yes	No
Geetha and Udayakumar (2016)	Yes	Yes	No	No	Yes
Hossen et al. (2016)	Yes	Yes	No	Yes	Yes
Pervin et al.(2016)	No	Yes	No	Yes	Yes
Shaikh et al. (2017)	Yes	No	No	No	No
Pervin et al. (2018)	No	Yes	No	Yes	Yes
Mashud et al. (2018a)	Yes	No	No	No	Yes
Mashud et al. (2018b)	Yes	No	No	Yes	No
Mashud et al. (2019)	Yes	No	Yes	No	Yes
This paper	Yes	Yes	Yes	Yes	Yes

This work is proposed on different combination of parameters used in traditional EOQ model that is unique in this research phenomena. It is very realistic that the demand depend mainly on three factors those are
(i) Selling price: It should be in range of buying capability of the targeted customers to increase demand and demand will be fall if the selling price gone out of range.
(ii) Time: Demand will vary with time elapse according as the need and choice of targeted consumers. It can be rise or fall with time.
(iii) Advertisement: It is one of the best policy to rise demand for a business unit after the quality of inventories. Customers always not aware about all products in market. They know the quality and utility of products by different types of advertisement.

The main focus in this paper is the reduction of cost and increase demand by using an EOQ model with the above mentioned combination of instantaneous deteriorating items when it is partially backlogged.

2. Problem Description

In this study, we described an inventory model based on price, continuous time, Number of advertisement per unit time., constant deterioration rate and partially backlogged shortage. Recently, Promotion of product through TV, radio, Billboard are very useful to stimulate the customer's demand.Practically, retailers always want to finish their stock fast to get more profit. So, to attract more customers, Retailer needs to apply some policies like discount on selling price and promotion of products to increase the customer's interest. And to have better results they spend some money in advertisement. Therefore, in this model, we considered the demand function dependent on price, continuous time and frequency of advertisement. In shortage time, the customers demand is partially backlogged with waiting time (T-t). The deterioration is considered instantaneous i.e., when the item stock is in retailer's house, deterioration will immediately start with a constant rate. Inventories S will be depleted into two way, one is customers demand and another one is deterioration of products. The corresponding inventory problem constitutes a constraint optimization problem. Here, we solved this problem by using Lingo 15 software, represented in a graph the help of Mathlab2017a. Finally, to illustrate and validate the inventory model, we used a numerical example considering fixed price. A sensitivity analysis is carried out to study the effect of changes of different inventory parameters changing one parameter at a time and keeping the other parameters constant.

3. Assumption and Notations

The mathematical model proposed in this paper is based on following assumptions.

(i) The replenishment rate is infinite.
(ii) Lead-time is negligible.
(iii) The planning horizon of the inventory system is infinite.
(iv) The demand function is,

$$D(t)=\begin{cases} Q^{\gamma}(a-bp+ct) & when\ I(t)\geq 0 \\ -\dfrac{D}{1+\delta(T-t)} & when\ I(t)<0 \end{cases}$$

(v) I1 (t) denotes the inventory level at any time $t\in[0,\ t_d]$, with the deterioration of product. I2(t) stands for the inventory level at any time $t\in[t_d\,,T]$, without the product deterioration.

(vi) Here, we allowed shortages and this shortage are backlogged with the rate of $\frac{1}{1+\delta(T-t)}$, where δ is the backlogging parameter. (T-t) is the waiting time for the customer to receive products from the retailer.

3.1. Notations

In addition, the following nomenclatures are used throughout the paper development.

Notations	Units	Description
c_o	\$/unit	Replenishment cost per order
c_p	\$/unit	Purchasing cost per unit
c_h	\$/unit	Holding cost per unit per unit time
c_a	\$/unit	Advertisement cost per frequency
c_b	\$/unit	Shortage cost per unit per unit time
θ	constant	Deterioration rate
S	units	Maximum stock per cycle
R	units	Maximum shortage per cycle
M	units	Total order quantity
$I(t)$	units	Inventory level at any time t where $0\leq t\leq T$
$TC\ (t_d\,, T)$	\$/year	The total cost per unit time
		Decision variables
t_d	year	Time at which deterioration occurs.
T	year	The length of the replenishment cycle.

4. Formulation of proposed inventory model.

Based on the above-mentioned assumptions and notations, we built an inventory model which is depicted in the following figure 1.

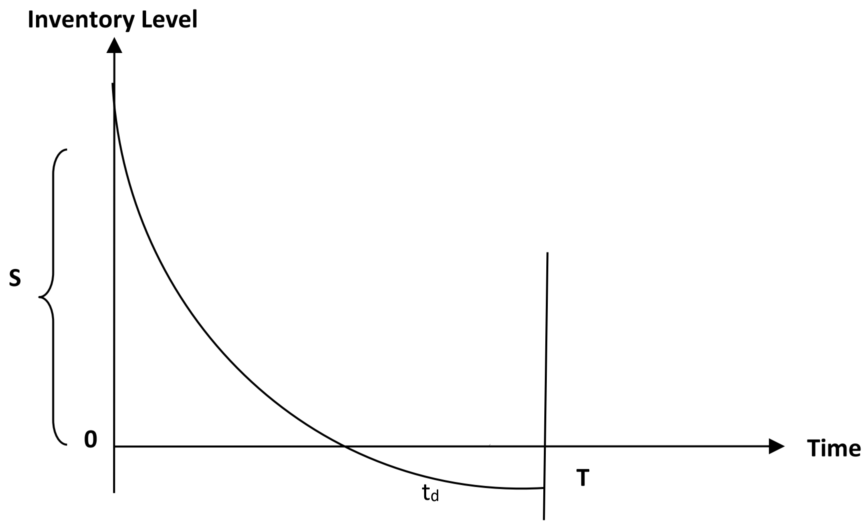

Figure 1. Graphical representation of the inventory system: inventory level vs. time.

Initially, a retailer purchased goods (*S+R)* units to meet the customers' demands. This stock was depleted due to customers demand as well as deterioration. As a result, at time $t=t_d$, the stock will become zero. After time t_d , the shortages will be continued till the start of next replenishment cycle appear up-to the total cycle time *T*. Therefore, the inventory system is govern by the following differential equation with considering demand $D(t)=Q^{\gamma}(a-bp+ct)$

$$\frac{dI_1(t)}{dt}+\theta I_1(t)=-D,\ 0\le t\le t_d \tag{1}$$

with the condition $I_1(t)=S\ at\ t=0$,

$$\frac{dI_2(t)}{dt}=-\frac{D}{1+\delta(T-t)},\ t_d\le t\le T \tag{2}$$

with $I_2(t)=-R\ at\ t=T$, $I_1(t), I_2(t)\, is\, continuous\ at\ t=t_d$

5. Solution procedures of differential equations from 1-2

With the help of condition $I_1(t)=0\ at\ t=t_d$ after solving equation (1) we get:

$$I_1(t)=\frac{Q^{\gamma}(a-bp)}{\theta}\left[e^{\theta(t_d-t)}-1\right]+\frac{Q^{\gamma}c}{\theta}\left[(t_d-\frac{1}{\theta})e^{\theta(t_d-t)}-(t-\frac{1}{\theta})\right];\ 0\le t\le t_d \tag{3}$$

Also, by applying the condition $I_1(t)=S\ at\ t=0$ in equation (3) we get the initial stock

$$S=\frac{Q^{\gamma}(a-bp)}{\theta}\left[e^{\theta t_d}-1\right]+\frac{Q^{\gamma}c}{\theta}\left[(t_d-\frac{1}{\theta})e^{\theta t_d}+\frac{1}{\theta})\right] \tag{4}$$

With the help of condition $I_2(t)=0\ at\ t=t_d$ after solving equation (2) we get:

$$I_2(t)=N\log\frac{(1+\delta(T-t))}{(1+\delta(T-t_d))}+\frac{Q^{\gamma}c}{\delta}(t-t_d)\quad ,\ t_d\le t\le T \tag{5}$$

Where,

$$N=\left[\frac{Q^{\gamma}(a-bp)}{\delta}+\frac{TQ^{\gamma}c}{\delta}+\frac{Q^{\gamma}c}{\delta^2}\right]$$

Also, by applying the condition $I_2(t)=-R\ at\ t=T$ in equation (5) we get the shortage

$$R=N\log(1+\delta(T-t_d))-\frac{Q^{\gamma}c}{\delta}(T-t_d) \tag{6}$$

Total order quantity

M= S+R

$$=\begin{bmatrix} \frac{Q^{\gamma}(a-bp)}{\theta}\left[e^{\theta t_d}-1\right]+\frac{Q^{\gamma}c}{\theta}\left[(t_d-\frac{1}{\theta})e^{\theta t_d}+\frac{1}{\theta})\right] \\ +N\log(1+\delta(T-t_d))-\frac{Q^{\gamma}c}{\delta}(T-t_d) \end{bmatrix} \tag{7}$$

6.1. The inventory-related total cost per unit time consists of the following components

a) Ordering cost per cycle = c_0

b) Holding cost per cycle (HC) $=c_h\int_0^{t_d} I_1(t)dt$

i.e.

$$\begin{bmatrix} -\left[\frac{c_hQ^{\gamma}(a-bp)}{\theta^2}+\frac{c_hQ^{\gamma}c}{\theta^2}(t_d-\frac{1}{\theta})\right](1-e^{\theta t_d}) \\ +\left[\frac{c_hQ^{\gamma}c}{\theta^2}-\frac{c_hQ^{\gamma}(a-bp)}{\theta}\right]t_d-\frac{c_hQ^{\gamma}ct_d^{\,2}}{2\theta} \end{bmatrix}$$

c) The purchase cost per cycle (PC)= c_p*M

i.e. $$c_p\begin{bmatrix} \frac{Q^{\gamma}(a-bp)}{\theta}\left[e^{\theta t_d}-1\right]+\frac{Q^{\gamma}c}{\theta}\left[(t_d-\frac{1}{\theta})e^{\theta t_d}+\frac{1}{\theta})\right] \\ +N\log(1+\delta(T-t_d))-\frac{Q^{\gamma}c}{\delta}(T-t_d) \end{bmatrix} \tag{8}$$

c) Shortage Cost (SC) = $c_b\int_{t_d}^{T}\left[-I_2(t)\right]dt$

i.e. $-\frac{Nc_b}{\delta}\log\left[1+\delta(T-t_d)\right]+Nc_b(T-t_d)+\frac{c_bQ^{\gamma}c}{\delta}\left[Tt_d-\frac{t_d^{\,2}}{2}-\frac{T^2}{2}\right]$

d) Advertisement cost (AC)= $Q^{\gamma}c_aT$

[For the values of HC, SC, PC see Appendix A.]

Therefore, total inventory cost*(X)=<ordering cost>+<purchase cost>+<holding cost>+<shortage cost>+<advertisement cost>*

i.e., $X(t_d,T)=OC+PC+HC+SC+AC$ (9)

Thus, the corresponding constrained optimization problem can be written as follows:

$$TC\left(t_d,T\right)=\frac{X}{T} \tag{10}$$

6.2. Problem: Minimize

Subject to $0\le t\le T$

The above mention problems can be solved by using the well-known generalized reduced gradient (GRG) method.

6.3. Algorithms

We solved the above-mentioned problem using the following steps:

Step 1: Input the value of all required parameters of the proposed inventory model.

Step 2: Solve the above discussed constrained optimization Problem and store the optimal value of TC^*,S^*,R^*,t_d^* and T^*.

Step 3: To check the optimality find the second order derivative with respect to both decision variables from equation (9).

Step 4: Check $\alpha>0,\ \beta>0$ and $\alpha\beta-\lambda^2>0$, where, $\alpha=\frac{\partial^2TC}{\partial t_d^{\,2}}>0,\ \beta=\frac{\partial^2TC}{\partial T^2}>0,\ \lambda=\frac{\partial^2TC}{\partial t_d\partial T}$.

Stop 5: If Step 4 is correct then Go to Step 6. Otherwise, if the hessian matrix $\alpha\beta-\lambda^2<0$ then the problem is infeasible.

Stop 6: Stop.

6.4. Theoretical Results

For the minimization of total cost function with respect to t_d the necessary condition is $\dfrac{\partial TC}{\partial t_d}=0$

$$\frac{1}{T}\left[c_p\left(\frac{\partial S}{\partial t_d}+\frac{\partial R}{\partial t_d}\right)+\frac{\partial HC}{\partial t_d}+\frac{\partial SC}{\partial t_d}+\frac{\partial OC}{\partial t_d}+\frac{\partial AC}{\partial t_d}\right]=0 \tag{11}$$

Now, differentiate equation (10) with respect to decision variable T,

$$\frac{\partial TC\left(t_d,T\right)}{\partial T}=\frac{T\dfrac{\partial X}{\partial T}-X}{T^2} \tag{12}$$

For the minimization of total cost function with respect to T, the necessary condition is $\dfrac{\partial TC}{\partial T}=\dfrac{\partial}{\partial T}\left(\dfrac{X}{T}\right)=0$ (13)

where, $\dfrac{\partial X}{\partial T}=c_p\left(\dfrac{\partial S}{\partial T}+\dfrac{\partial R}{\partial T}\right)+\dfrac{\partial HC}{\partial T}+\dfrac{\partial SC}{\partial T}+\dfrac{\partial OC}{\partial T}+\dfrac{\partial AC}{\partial T}$

Differentiating equation (11) with respect to t_d, we get

$$\alpha=\frac{\partial^2 TC\left(t_2,T\right)}{\partial t_d^2}=\frac{1}{T}\frac{\partial^2 X}{\partial t_d^2} \tag{14}$$

Differentiating equation (12) with respect to t_d,

$$\lambda=\frac{\partial^2 TC\left(t_d,T\right)}{\partial T\partial t_d}=-\frac{1}{T^2}\frac{\partial X}{\partial t_d}+\frac{1}{T}\frac{\partial^2 X}{\partial T\partial t_d} \tag{15}$$

Differentiate equation (12) with respect to T,

$$\begin{aligned}\beta=\frac{\partial^2 TC\left(t_2,T\right)}{\partial T^2}&=\frac{T^2\left(\dfrac{\partial^2 X}{\partial T^2}-\dfrac{\partial X}{\partial T}\right)-2T\left(T\dfrac{\partial X}{\partial T}-X\right)}{T^4}\\&=\frac{T\left(\dfrac{\partial^2 X}{\partial T^2}-\dfrac{\partial X}{\partial T}\right)-2\left(T\dfrac{\partial X}{\partial T}-X\right)}{T^3}\\&=\frac{T\dfrac{\partial^2 X}{\partial T^2}-2T\dfrac{\partial X}{\partial T}+2X}{T^3}\end{aligned} \tag{16}$$

[For details see *appendix A*]

Theorem: The total cost function $TC\left(t_d,T\right)$ is strictly convex in both decision variables $\left(t_d,T\right)$ and there exist unique optimal solutions $\left(t_d^*,T^*\right)$ if $\alpha>0,\ \beta>0$ and $\alpha\beta-\lambda^2>0.$

Proof: See the Appendix B.

6.5. Numerical Illustrations

To illustrate the developed model, a numerical example with the following values of different parameters has been considered.

Let us take the following parameters value in appropriate units

C_0=\$500/ Order, $a=400$; $b=0.1$; $p=6$; $c=2$; $c_p=4$ per unit per unit time, $c_h=1$ per unit per unit time, $c_b=15$ per unit per unit time $c_a=3$ per unit time, $\gamma=0.8$, $\delta=1.5$, $\theta=0.5$, $Q=1$;

The values of parameters considered here are realistic, though these values are not taken from any case study of an existing inventory system. The computational work has been performed on a PC with Intel core i3 computer of 2.40 GHz Processor.

According to the proposed algorithm, the optimal solution was obtained with the help of GRG method. The optimum values of t_d, T, S and R along with minimum average cost are displayed in Table1.

Table 1. Optimal values of different decision variables

Decision Variables	**Values**
TC	2549.066
S	312.3849
R	132.3715
t_d	0.6591658
T	1.086091

6.6. Convexity of Cost function

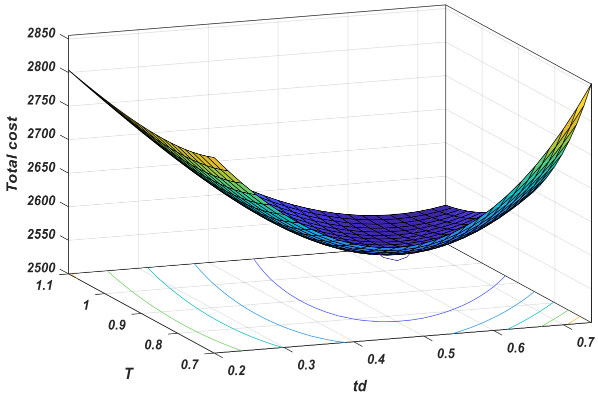

Figure 2. Total cost vs. T vs. td

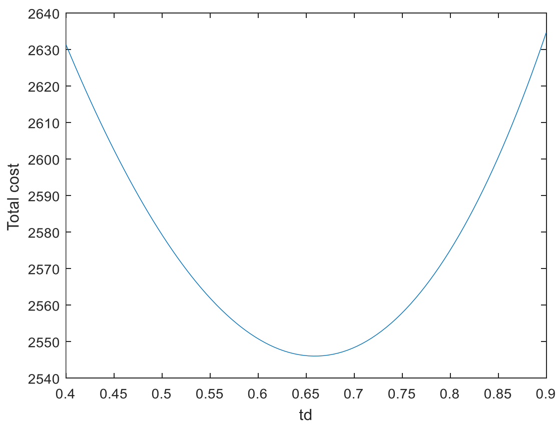

Fig 2.1. Total cost vs. td

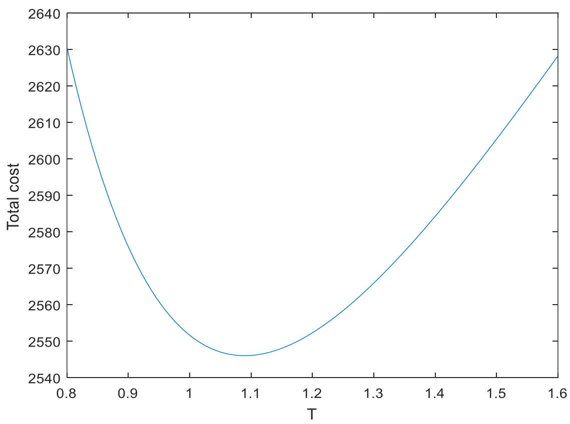

Figure 2.2. Total cost vs. T

7. Sensitivity Analysis

The above numerical example is used to study the effect of under- or over-estimation of the inventory system parameters on the optimal values of the initial stock level, highest shortage level, cycle length, and the minimum total cost of the system. The percentage changes in the above-mentioned optimal values are taken as measures of sensitivity. The analysis is carried out by changing (increasing and decreasing) the parameters by -20% to +20%. The results are obtained by changing one parameter at a time and keeping the other parameters at their original values. The results of these analyses are given in Tables 2.

Table 2. Sensitivity analysis with respect to different parameters

Parameter	% changes of parameters	Changes in TC^*	Changes in			
			S^*	R^*	t_d^*	T^*
c_0	– 20	2451.280	280.3114	115.3079	0.6007458	0.9602922
	– 10	2501.675	296.8468	123.9682	0.6310793	1.024294
	10	2593.870	327.0635	140.5616	0.6853377	1.146135
	20	2636.412	340.9897	148.5722	0.7098525	1.204779
δ	– 20	2584.035	324.2857	112.6265	0.6804115	1.014293
	– 10	2567.890	318.7973	121.7347	0.6706418	1.045711
	10	2526.920	304.8197	144.9243	0.6455407	1.139358
	20	2500.626	295.8044	159.9028	0.6291807	1.211801
θ	– 20	2503.346	342.7445	124.0961	0.7365138	1.129977
	– 10	2527.147	326.8218	128.3734	0.6956073	1.106236
	10	2569.309	299.2212	136.1158	0.6264679	1.068878
	20	2588.053	287.1594	139.6286	0.5969445	1.054081
γ	– 20	2549.066	312.3849	132.3715	0.6591658	1.086091
	– 10	2549.066	312.3849	132.3715	0.6591658	1.086091
	10	2549.066	312.3849	132.3715	0.6591658	1.086091
	20	2549.066	312.3849	132.3715	0.6591658	1.086091
P	– 20	2549.691	312.4340	132.3866	0.6590857	1.085911
	– 10	2549.379	312.4094	132.3791	0.6591258	1.086001
	10	2548.754	312.3604	132.3640	0.6592059	1.086181
	20	2548.441	312.3358	132.3565	0.6592460	1.086271
c_b	– 20	2447.767	277.8476	196.9322	0.5961861	1.321107
	– 10	2507.041	298.0937	158.5016	0.6333479	1.172371
	10	2580.206	322.9462	113.5248	0.6780314	1.030597
	20	2604.117	331.0411	99.31950	0.6923707	0.9922356
c_h	– 20	2530.454	328.0236	128.9776	0.6870377	1.100122
	– 10	2539.934	319.9954	130.7011	0.6727785	1.092869
	10	2557.869	305.1582	133.9914	0.6461524	1.079746
	20	2566.360	298.2850	135.5630	0.6336957	1.073799
c_p	– 20	2212.009	354.4506	110.6037	0.7332631	1.074655
	– 10	2382.859	332.7573	120.9726	0.6953975	1.076669
	10	2710.354	292.9899	144.9941	0.6240455	1.103835
	20	2866.358	274.2822	159.0781	0.5895690	1.131344
a	– 20	2126.850	277.6874	121.8252	0.7204679	1.230960
	– 10	2339.442	295.5635	127.2292	0.6876902	1.151657
	10	2756.175	328.3200	137.2873	0.6340453	1.030700
	20	2961.115	343.4965	142.0040	0.6116974	0.9830916
b	– 20	2549.691	312.4340	132.3866	0.6590857	1.085911
	– 10	2549.379	312.4094	132.3791	0.6591258	1.086001
	10	2548.754	312.3604	132.3640	0.6592059	1.086181
	20	2548.441	312.3358	132.3565	0.6592460	1.086271
C	– 20	2547.902	312.7446	132.5112	0.6600067	1.087982
	– 10	2548.485	312.5645	132.4412	0.6595856	1.087035
	10	2549.647	312.2059	132.3021	0.6587472	1.085151
	20	2550.228	312.0275	132.2329	0.6583299	1.084214
C_a	– 20	2548.466	312.3849	132.3715	0.6591658	1.086091
	– 10	2548.766	312.3849	132.3715	0.6591658	1.086091
	10	2549.366	312.3849	132.3715	0.6591658	1.086091
	20	2549.666	312.3849	132.3715	0.6591658	1.086091
Q	– 20	2203.805	284.7234	123.9797	0.7087678	1.202131
	– 10	2379.348	299.0373	128.3011	0.6822745	1.138975
	10	2713.778	324.9226	136.2263	0.6387505	1.040918
	20	2874.117	336.7694	139.8931	0.6205226	1.001714

Form Table 2, the following observations can be made:

(i) The effects on total cost with respect to the value of the parameter $c_0, \theta, c_b, c_h, a, c_a, c, Q, c_p$ is proportional, while for the others parameters they are inversely proportional. And, there is no change in total cost with respect to γ.

(ii) The effects on initial on-hand stock with respect to the value of the parameter $c_0,\ c_b,\ a, Q$ are proportional, while for the other parameters they are inversely proportional. And, there is no change in total cost with respect to γ and C_a.

(iii) The effects on maximum shortages with respect to the value of the parameter $c_0, \delta, \theta, c_h, c_p, a, Q$ are proportional, while for the other parameters they are inversely proportional. And, there is no change in total cost with respect to γ, c_a.

(iv) The effects on decision variables t_d with respect to the value of the parameter c_0, p, c_b, b are proportional, while for the other parameters they are inversely proportional. And, there is no change in total cost with respect to γ, c_a.

(v) The effects on total cost with respect to the value of the parameter c_0, δ, p, c_p, b are proportional, while for the other parameters they are inversely proportional. And, there is no change in total cost with respect to γ, c_a.

Special cases:

(i) If $Q^\gamma = 1, c = 0$ and $\delta = 0$, then our model reduces to an inventory model for deteriorating items with shortage and selling price-dependent demand of *K. Das et. all (1999).*

(ii) If $\delta = 0$, then the EOQ model is fully backlogged like *Shaikh et al. (2017).*

(iii) If $p = 0$, then our demand function depends on time and advertisement frequency.

(iv) If $\delta \to \infty$, then there is no shortage and our model becomes an EOQ model with no shortage.

(v) If $c = 0$, then demand depends on selling price and advertisement frequency.

(vi) If $Q^\gamma = 1$, $p = 0$ and $c = 0$, then our model becomes an EOQ model with constant demand.

(vii) If $Q^\gamma = 1$ and $c = 0$,our EOQ model reduce to Fuzzy EOQ model for deteriorating items with price-dependent demand and time-varying holding cost By *Bhunia et. all (2012).*

8. Concluding Remarks

In this work, we described an inventory model according to selling price, continuous time and frequency of advertisement depending on demand with partially backlogged shortages. The stock were depleted due not only to customers demand but also to deterioration. We found the length where deterioration occurred and also found the optimal cycle length in order to minimize the total cost. The corresponding inventory problem creates a nonlinear constraint optimization problem. We solved this problem by the help of GRG method and validated it by sketching a convex graph of the cost function and also making a sensitivity table to show the validate range for the parameters, which makes the cost function more perfection. One can easily extend this model by taking trade credit policy both single and two level. It is also possible to modify the model by using preservation technology. One can also extend it under fuzzy environment and under inflationary environment.

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Appendix A:

The values are

$$\begin{aligned}&\frac{dI_1(t)}{dt}+\theta I_1(t)=-Q^{\gamma}\left[(a-bp)+ct\right]\ ;\\ &I_1(t_d)=0\,,I_1(0)=s\ ,0\le t\le t_d\\ &IF=e^{\int\theta dt}=e^{\theta t}\end{aligned} \qquad (A.1)$$

$$(A.1)\Rightarrow$$
$$I_1(t)e^{\theta t}=-\int Q^{\gamma}\left[(a-bp)+ct\right]e^{\theta t}dt$$
$$\Rightarrow I_1(t)e^{\theta t}=-Q^{\gamma}(a-bp)\int e^{\theta t}dt-Q^{\gamma}c\int te^{\theta t}dt$$
$$\Rightarrow I_1(t)e^{\theta t}=-Q^{\gamma}(a-bp)\frac{e^{\theta t}}{\theta}-Q^{\gamma}c\left[\frac{te^{\theta t}}{\theta}-\frac{1}{\theta}\int e^{\theta t}dt\right]$$
$$\Rightarrow I_1(t)e^{\theta t}=-Q^{\gamma}(a-bp)\frac{e^{\theta t}}{\theta}-Q^{\gamma}c\left[\frac{te^{\theta t}}{\theta}-\frac{e^{\theta t}}{\theta^2}\right]+c_1$$
$$\Rightarrow I_1(t)e^{\theta t}=-Q^{\gamma}(a-bp)\frac{e^{\theta t}}{\theta}-Q^{\gamma}c\frac{e^{\theta t}}{\theta}\left[t-\frac{1}{\theta}\right]+c_1$$
$$\Rightarrow I_1(t)=-\frac{Q^{\gamma}(a-bp)}{\theta}-\frac{Q^{\gamma}c}{\theta}\left[t-\frac{1}{\theta}\right]+c_1e^{-\theta t}$$

$$Apply\ I_1(t_d)=0\,in\,(A.1)\Rightarrow$$
$$0=-\frac{Q^{\gamma}(a-bp)}{\theta}-\frac{Q^{\gamma}c}{\theta}\left[t_d-\frac{1}{\theta}\right]+c_1e^{-\theta t_d}\qquad\text{(A.2)}$$
$$\Rightarrow c_1=\frac{Q^{\gamma}(a-bp)e^{\theta t_d}}{\theta}+\frac{Q^{\gamma}c}{\theta}\left[t_d-\frac{1}{\theta}\right]e^{\theta t_d}$$

$$Now(A.2)\Rightarrow$$
$$I_1(t)=-\frac{Q^{\gamma}(a-bp)}{\theta}-\frac{Q^{\gamma}c}{\theta}\left[t-\frac{1}{\theta}\right]+\frac{Q^{\gamma}(a-bp)e^{\theta(t_d-t)}}{\theta}+\frac{Q^{\gamma}c}{\theta}\left[t_d-\frac{1}{\theta}\right]e^{\theta(t_d-t)}$$
$$\Rightarrow I_1(t)=\frac{Q^{\gamma}(a-bp)}{\theta}\left[e^{\theta(t_d-t)}-1\right]+\frac{Q^{\gamma}c}{\theta}\left[(t_d-\frac{1}{\theta})e^{\theta(t_d-t)}-(t-\frac{1}{\theta})\right]\qquad\text{(A.3), (A.4)}$$
$$Apply\ I_1(0)=s\,in\,(3)\Rightarrow$$
$$s=\frac{Q^{\gamma}(a-bp)}{\theta}\left[e^{\theta t_d}-1\right]+\frac{Q^{\gamma}c}{\theta}\left[(t_d-\frac{1}{\theta})e^{\theta t_d}+\frac{1}{\theta})\right]$$

Now
$$\frac{dI_2(t)}{dt}=-\frac{Q^{\gamma}(a-bp)}{1+\delta(T-t)}-\frac{Q^{\gamma}ct}{1+\delta(T-t)}\ ;I_2(t_d)=0,I_2(0)=-R\ ,t_d\le t<T\qquad\text{(A.5)}$$
$$\Rightarrow\int dI_2(t)dt=-Q^{\gamma}(a-bp)\int\frac{dt}{1+\delta(T-t)}-Q^{\gamma}c\int\frac{tdt}{1+\delta(T-t)}$$
$$put,\,1+\delta(T-t)=z\Rightarrow dt=-\frac{dz}{\delta}\,and\,t=T-\frac{z}{\delta}+\frac{1}{\delta}$$

Then
$$I_2(t)=\frac{Q^{\gamma}(a-bp)}{\delta}\log(z)+\frac{Q^{\gamma}c}{\delta}\int\frac{(T-\frac{z}{\delta}+\frac{1}{\delta})dz}{z}$$
$$\Rightarrow I_2(t)=\frac{Q^{\gamma}(a-bp)}{\delta}\log(z)+\frac{Q^{\gamma}c}{\delta}\left[T\log(z)-\frac{z}{\delta}+\frac{\log(z)}{\delta}\right]+c_2$$
$$\Rightarrow I_2(t)=\left[\frac{Q^{\gamma}(a-bp)}{\delta}+\frac{TQ^{\gamma}c}{\delta}+\frac{Q^{\gamma}c}{\delta^2}\right]\log(z)-\frac{Q^{\gamma}c}{\delta^2}z+c_2$$
$$\Rightarrow I_2(t)=\left[\frac{Q^{\gamma}(a-bp)}{\delta}+\frac{TQ^{\gamma}c}{\delta}+\frac{Q^{\gamma}c}{\delta^2}\right]\log(1+\delta(T-t))-\frac{Q^{\gamma}c}{\delta^2}(1+\delta(T-t))+c_2$$
$$\Rightarrow I_2(t)=N\log(1+\delta(T-t))-\frac{Q^{\gamma}c}{\delta^2}(1+\delta(T-t))+c_2$$
$$where\ N=\left[\frac{Q^{\gamma}(a-bp)}{\delta}+\frac{TQ^{\gamma}c}{\delta}+\frac{Q^{\gamma}c}{\delta^2}\right]$$

(A.6)

$$Apply\ I_2(t_d)=0\,in\,(A.6)\Rightarrow$$

$$0 = N\log(1+\delta(T-t_d)) - \frac{Q^\gamma c}{\delta^2}(1+\delta(T-t_d)) + c_2$$

$$\Rightarrow c_2 = \frac{Q^\gamma c}{\delta^2}(1+\delta(T-t_d)) - N\log(1+\delta(T-t_d))$$

$$put\, c_2\, in\,(A.6) \Rightarrow$$

$$I_2(t) = N\log(1+\delta(T-t)) - \frac{Q^\gamma c}{\delta^2}(1+\delta(T-t)) + \frac{Q^\gamma c}{\delta^2}(1+\delta(T-t_d)) - N\log(1+\delta(T-t_d)) \qquad \text{(A.7)}$$

$$\Rightarrow I_2(t) = N\log\frac{(1+\delta(T-t))}{(1+\delta(T-t_d))} + \frac{Q^\gamma c}{\delta^2}(1+\delta T-\delta t_d - 1 - \delta T + \delta t)$$

$$\Rightarrow I_2(t) = N\log\frac{(1+\delta(T-t))}{(1+\delta(T-t_d))} + \frac{Q^\gamma c}{\delta}(t-t_d)$$

$$Apply\ I_2(T) = -R\, in\,(7) \Rightarrow$$

$$-R = N\log\frac{1}{(1+\delta(T-t_d))} + \frac{Q^\gamma c}{\delta}(T-t_d)$$

$$\Rightarrow R = N\log(1+\delta(T-t_d)) - \frac{Q^\gamma c}{\delta}(T-t_d)$$

$$Now\ M = S + R$$

$$\Rightarrow M = \frac{Q^\gamma (a-bp)}{\theta}\left[e^{\theta t_d} - 1\right] + \frac{Q^\gamma c}{\theta}\left[(t_d - \frac{1}{\theta})e^{\theta t_d} + \frac{1}{\theta})\right] + N\log(1+\delta(T-t_d)) - \frac{Q^\gamma c}{\delta}(T-t_d)$$

$$Purchas\ \cos t\ PC = Mc_p$$

$$\Rightarrow PC = c_p\left[\frac{Q^\gamma (a-bp)}{\theta}\left[e^{\theta t_d} - 1\right] + \frac{Q^\gamma c}{\theta}\left[(t_d - \frac{1}{\theta})e^{\theta t_d} + \frac{1}{\theta})\right] + N\log(1+\delta(T-t_d)) - \frac{Q^\gamma c}{\delta}(T-t_d)\right]$$

$$Now\, holding\, \cos t:$$

$$HC = c_h\int_0^{t_d} I_1(t)dt$$

$$\Rightarrow HC = \frac{c_h Q^\gamma (a-bp)}{\theta}\left[\int_0^{t_d} e^{\theta(t_d - t)}dt - \int_0^{t_d} dt\right] + \frac{c_h Q^\gamma c}{\theta}\int_0^{t_d}\left[(t_d - \frac{1}{\theta})e^{\theta(t_d-t)} - (t - \frac{1}{\theta})\right]dt$$

$$\Rightarrow HC = \left[\frac{c_h Q^\gamma (a-bp)}{\theta} + \frac{c_h Q^\gamma c}{\theta}(t_d - \frac{1}{\theta})\right]\int_0^{t_d} e^{\theta(t_d-t)}dt - \frac{c_h Q^\gamma (a-bp)}{\theta}\int_0^{t_d} dt - \frac{c_h Q^\gamma c}{\theta}\int_0^{t_d}(t - \frac{1}{\theta})dt$$

$$put\ \theta(t_d - t) = L \Rightarrow dt = -\frac{dL}{\theta}\, in\, 1'st\, \text{int}\, egral \qquad \text{(A.8), (A.9)}$$

$$\Rightarrow HC = -\left[\frac{c_h Q^\gamma (a-bp)}{\theta^2} + \frac{c_h Q^\gamma c}{\theta^2}(t_d - \frac{1}{\theta})\right]\int_0^{t_d} e^L dL - \frac{c_h Q^\gamma (a-bp)}{\theta}\left[t\right]_0^{t_d} - \frac{c_h Q^\gamma c}{\theta}\left[\frac{t^2}{2} - \frac{t}{\theta}\right]_0^{t_d}$$

$$\Rightarrow HC = -\left[\frac{c_h Q^\gamma (a-bp)}{\theta^2} + \frac{c_h Q^\gamma c}{\theta^2}(t_d - \frac{1}{\theta})\right]\left[e^{\theta(t_d-t)}\right]_0^{t_d} - \frac{c_h Q^\gamma (a-bp)}{\theta}t_d - \frac{c_h Q^\gamma c t_d^{\,2}}{2\theta} + \frac{c_h Q^\gamma c t_d}{\theta^2}$$

$$\Rightarrow HC = -\left[\frac{c_h Q^\gamma (a-bp)}{\theta^2} + \frac{c_h Q^\gamma c}{\theta^2}(t_d - \frac{1}{\theta})\right](1 - e^{\theta t_d}) + \left[\frac{c_h Q^\gamma c}{\theta^2} - \frac{c_h Q^\gamma (a-bp)}{\theta}\right]t_d - \frac{c_h Q^\gamma c t_d^{\,2}}{2\theta} \qquad \text{(A.10)}$$

Then

$$SC = Nc_b \log\left[1+\delta(T-t_d)\right]t_d - Nc_b \log\left[1+\delta(T-t_d)\right](t_d - T)$$

$$+\frac{c_b Q^{\gamma} c}{\delta}\left[Tt_d - \frac{t_d^{\,2}}{2} - \frac{T^2}{2}\right] + Nc_b(T-t_d) - Nc_b(T+\frac{1}{\delta})\log(1+\delta(T-t_d))$$

$$\Rightarrow SC = \left[Nc_b t_d - Nc_b(t_d - T) - Nc_b(T+\frac{1}{\delta})\right]\log\left[1+\delta(T-t_d)\right] + Nc_b(T-t_d) + \frac{c_b Q^{\gamma} c}{\delta}\left[Tt_d - \frac{t_d^{\,2}}{2} - \frac{T^2}{2}\right]$$

$$\Rightarrow SC = -\frac{Nc_b}{\delta}\log\left[1+\delta(T-t_d)\right] + Nc_b(T-t_d) + \frac{c_b Q^{\gamma} c}{\delta}\left[Tt_d - \frac{t_d^{\,2}}{2} - \frac{T^2}{2}\right]$$

Now Total cost :

$$X(t_d, T) = OC + HC + SC + PC + AC$$

$$TC = \frac{X}{T}$$

$$PC = c_p\left[\frac{Q^{\gamma}(a-bp)}{\theta}\left[e^{\theta t_d} - 1\right] + \frac{Q^{\gamma} c}{\theta}\left[(t_d - \frac{1}{\theta})e^{\theta t_d} + \frac{1}{\theta})\right] + N\log(1+\delta(T-t_d)) - \frac{Q^{\gamma} c}{\delta}(T-t_d)\right]$$

Now

$$PC_T = \frac{\partial}{\partial T}(PC)$$

$$\Rightarrow PC_T = 0 + 0 + c_p N_T \log(1+\delta(T-t_d)) + \frac{N\delta c_p}{1+\delta(T-t_d)} - \frac{Q^{\gamma} c c_p}{\delta} \quad ;$$

$$where\ N = \left[\frac{Q^{\gamma}(a-bp)}{\delta} + \frac{TQ^{\gamma} c}{\delta} + \frac{Q^{\gamma} c}{\delta^2}\right]$$

$$then \quad N_T = \frac{Q^{\gamma} c}{\delta}$$

$$\Rightarrow PC_T = c_p N_T \log(1+\delta(T-t_d)) + \frac{N\delta c_p}{1+\delta(T-t_d)} - N_T c_p$$

$$PC_{TT} = \frac{\partial}{\partial T}(PC_T)$$

$$\Rightarrow PC_{TT} = \frac{Q^\gamma \delta c c_p}{\delta\left[1+\delta(T-t_d)\right]} + \frac{\delta c_p N_T}{1+\delta(T-t_d)} + \frac{\delta^2 c_p N(-1)}{\left[1+\delta(T-t_d)\right]^2} - 0$$

$$\Rightarrow PC_{TT} = \frac{Q^\gamma c c_p}{\left[1+\delta(T-t_d)\right]} + \frac{\delta c_p N_T}{1+\delta(T-t_d)} - \frac{\delta^2 c_p N}{\left[1+\delta(T-t_d)\right]^2}$$

$$PC_{Tt_d} = \frac{\partial}{\partial t_d}(PC_T)$$

$$\Rightarrow PC_{Tt_d} = \frac{c_p N_T(-\delta)}{\left[1+\delta(T-t_d)\right]} + \frac{\delta c_p N(-\delta)(-1)}{\left[1+\delta(T-t_d)\right]^2} + 0$$

$$\Rightarrow PC_{Tt_d} = -\frac{c_p N_T \delta}{\left[1+\delta(T-t_d)\right]} + \frac{\delta c_p N \delta}{\left[1+\delta(T-t_d)\right]^2}$$

$$\Rightarrow PC_{Tt_d} = \frac{\delta^2 c_p N}{\left[1+\delta(T-t_d)\right]^2} - \frac{c_p N_T \delta}{\left[1+\delta(T-t_d)\right]}$$

$$PC_{t_d} = \frac{\partial}{\partial t_d}(PC)$$

$$\Rightarrow PC_{t_d} = \frac{c_p Q^\gamma (a-bp)}{\theta}\theta e^{\theta t_d} + \frac{c_p Q^\gamma c}{\theta} e^{\theta t_d} + \frac{c_p Q^\gamma c}{\theta}(t_d - \frac{1}{\theta})\theta e^{\theta t_d} + \frac{c_p N(-\delta)}{\left[1+\delta(T-t_d)\right]} - \frac{c_p Q^\gamma c}{\delta}(-1)$$

$$\Rightarrow PC_{t_d} = c_p Q^\gamma (a-bp) e^{\theta t_d} + \frac{c_p Q^\gamma c}{\theta} e^{\theta t_d} + c_p Q^\gamma c(t_d - \frac{1}{\theta}) e^{\theta t_d} - \frac{c_p N\delta}{\left[1+\delta(T-t_d)\right]} + c_p N_T$$

$$PC_{t_d t_d} = \frac{\partial}{\partial t_d}(PC_{t_d})$$

$$\Rightarrow PC_{t_d t_d} = c_p Q^\gamma (a-bp)\theta e^{\theta t_d} + \frac{c_p Q^\gamma c}{\theta}\theta e^{\theta t_d} + c_p Q^\gamma c(t_d - \frac{1}{\theta})\theta e^{\theta t_d} + c_p Q^\gamma c e^{\theta t_d} - \frac{c_p N\delta(-1)(-\delta)}{\left[1+\delta(T-t_d)\right]^2} + 0$$

$$\Rightarrow PC_{t_d t_d} = c_p Q^\gamma (a-bp)\theta e^{\theta t_d} + c_p Q^\gamma c e^{\theta t_d} + c_p Q^\gamma c(t_d - \frac{1}{\theta})\theta e^{\theta t_d} + c_p Q^\gamma c e^{\theta t_d} - \frac{c_p N\delta^2}{\left[1+\delta(T-t_d)\right]^2}$$

$$\Rightarrow PC_{t_d t_d} = c_p Q^\gamma (a-bp)\theta e^{\theta t_d} + 2c_p Q^\gamma c e^{\theta t_d} + c_p Q^\gamma c(t_d - \frac{1}{\theta})\theta e^{\theta t_d} - \frac{c_p N\delta^2}{\left[1+\delta(T-t_d)\right]^2}$$

Now

$$HC=-\left[\frac{c_hQ^\gamma(a-bp)}{\theta^2}+\frac{c_hQ^\gamma c}{\theta^2}(t_d-\frac{1}{\theta})\right](1-e^{\theta t_d})+\left[\frac{c_hQ^\gamma c}{\theta^2}-\frac{c_hQ^\gamma(a-bp)}{\theta}\right]t_d-\frac{c_hQ^\gamma ct_d{}^2}{2\theta}$$

$$HC_T=0$$

$$HC_{TT}=0$$

$$HC_{t_d}=-\frac{c_hQ^\gamma(a-bp)}{\theta^2}(-\theta e^{\theta t_d})-\frac{c_hQ^\gamma c}{\theta^2}(t_d-\frac{1}{\theta})(-\theta e^{\theta t_d})-\frac{c_hQ^\gamma c}{\theta^2}(1-e^{\theta t_d})$$

$$+\frac{c_hQ^\gamma c}{\theta^2}-\frac{c_hQ^\gamma(a-bp)}{\theta}-\frac{c_hQ^\gamma ct_d}{\theta}$$

$$\Rightarrow HC_{t_d}=\frac{c_hQ^\gamma(a-bp)}{\theta}e^{\theta t_d}+\frac{c_hQ^\gamma c}{\theta}(t_d-\frac{1}{\theta})e^{\theta t_d}-\frac{c_hQ^\gamma c}{\theta^2}(1-e^{\theta t_d})$$

$$+\frac{c_hQ^\gamma c}{\theta^2}-\frac{c_hQ^\gamma(a-bp)}{\theta}-\frac{c_hQ^\gamma ct_d}{\theta}$$

$$HC_{t_dt_d}=\frac{\partial}{\partial t_d}(HC_{t_d})$$

$$\Rightarrow HC_{t_dt_d}=\frac{c_hQ^\gamma(a-bp)}{\theta}\theta e^{\theta t_d}+\frac{c_hQ^\gamma c}{\theta}(t_d-\frac{1}{\theta})\theta e^{\theta t_d}+\frac{c_hQ^\gamma c}{\theta}e^{\theta t_d}$$

$$-\frac{c_hQ^\gamma c}{\theta^2}(-\theta e^{\theta t_d})+0-\frac{c_hQ^\gamma c}{\theta}$$

$$\Rightarrow HC_{t_dt_d}=c_hQ^\gamma(a-bp)e^{\theta t_d}+c_hQ^\gamma c(t_d-\frac{1}{\theta})e^{\theta t_d}+\frac{c_hQ^\gamma c}{\theta}e^{\theta t_d}$$

$$+\frac{c_hQ^\gamma c}{\theta}(e^{\theta t_d})-\frac{c_hQ^\gamma c}{\theta}$$

$$\Rightarrow HC_{t_dt_d}=c_hQ^\gamma e^{\theta t_d}\left[(a-bp)+c(t_d-\frac{1}{\theta})\right]+\frac{2c_hQ^\gamma c}{\theta}e^{\theta t_d}-\frac{c_hQ^\gamma c}{\theta}$$

Now

$$SC=-\frac{Nc_b}{\delta}\log\left[1+\delta(T-t_d)\right]+Nc_b(T-t_d)+\frac{c_bQ^\gamma c}{\delta}\left[Tt_d-\frac{t_d{}^2}{2}-\frac{T^2}{2}\right]$$

$$SC_T=\frac{\partial}{\partial T}(SC)$$

$$\Rightarrow SC_T=-\frac{N_Tc_b}{\delta}\log\left[1+\delta(T-t_d)\right]-\frac{Nc_b\delta}{\delta\left[1+\delta(T-t_d)\right]}+Nc_b+\frac{c_bQ^\gamma c}{\delta}\left[t_d-T\right]$$

$$\Rightarrow SC_T=-\frac{N_Tc_b}{\delta}\log\left[1+\delta(T-t_d)\right]-\frac{Nc_b}{\left[1+\delta(T-t_d)\right]}+Nc_b+c_bN_T\left[t_d-T\right]$$

$$SC_{TT}=\frac{\partial}{\partial T}(SC_T)$$

$$\Rightarrow SC_{TT}=-\frac{N_T\delta c_b}{[1+\delta(T-t_d)]\delta}-0-\frac{N_Tc_b}{[1+\delta(T-t_d)]}-\frac{Nc_b(-1)\delta}{[1+\delta(T-t_d)]^2}+N_Tc_b-c_bN_T$$

$$\Rightarrow SC_{TT}=-\frac{N_Tc_b}{[1+\delta(T-t_d)]}-\frac{N_Tc_b}{[1+\delta(T-t_d)]}+\frac{Nc_b\delta}{[1+\delta(T-t_d)]^2}$$

$$\Rightarrow SC_{TT}=\frac{Nc_b\delta}{[1+\delta(T-t_d)]^2}-\frac{2N_Tc_b}{[1+\delta(T-t_d)]}$$

$$SC_{Tt_d}=\frac{\partial}{\partial t_d}(SC_T)$$

$$\Rightarrow SC_{Tt_d}=-\frac{N_T(-\delta)c_b}{[1+\delta(T-t_d)]\delta}-\frac{Nc_b(-\delta)(-1)}{[1+\delta(T-t_d)]^2}+0+c_bN_T$$

$$\Rightarrow SC_{Tt_d}=\frac{N_Tc_b}{[1+\delta(T-t_d)]}-\frac{Nc_b\delta}{[1+\delta(T-t_d)]^2}+c_bN_T$$

$$SC_{t_d}=\frac{\partial}{\partial t_d}(SC)\Rightarrow SC_{t_d}=-\frac{(-\delta)Nc_b}{\delta[1+\delta(T-t_d)]}-Nc_b+c_bN_T[T-t_d]$$

$$\Rightarrow SC_{t_d}=\frac{Nc_b}{[1+\delta(T-t_d)]}-Nc_b+c_bN_T[T-t_d]$$

$$SC_{t_dt_d}=\frac{\partial}{\partial t_d}(SC_{t_d})$$

$$\Rightarrow SC_{t_dt_d}=\frac{(-\delta)(-1)Nc_b}{[1+\delta(T-t_d)]^2}-0-c_bN_T$$

$$\Rightarrow SC_{t_dt_d}=\frac{\delta Nc_b}{[1+\delta(T-t_d)]^2}-c_bN_T$$

$$TC(t_d,T)=\frac{1}{T}\left[\begin{array}{l}OC+\dfrac{Q^\gamma(a-bp)}{\theta}\left[e^{\theta t_d}-1\right](c_p+\dfrac{c_h}{\theta})+\dfrac{c_pQ^\gamma c}{\theta}\left[(t_d-\dfrac{1}{\theta})e^{\theta t_d}+\dfrac{1}{\theta})\right]\\ -\dfrac{Q^\gamma c}{\delta}\left[c_p(T-t_d)-c_b\left[Tt_d-\dfrac{t_d^{\ 2}}{2}-\dfrac{T^2}{2}\right]\right]+N\log[1+\delta(T-t_d)](c_p-\dfrac{c_b}{\delta})\\ +Nc_b(T-t_d)+\dfrac{c_hQ^\gamma c}{\theta^2}\left[t_d-(t_d-\dfrac{1}{\theta})(1-e^{\theta t_d})\right]-\dfrac{c_hQ^\gamma(a-bp)t_d}{\theta}-\dfrac{c_hQ^\gamma ct_d^{\ 2}}{2\theta}+Q^\gamma c_aT\end{array}\right]$$

$$\frac{\partial^2 TC}{\partial {t_d}^2}=\frac{1}{T}\left[\begin{array}{l} Q^\gamma (a-bp)e^{\theta t_d}\left(\theta c_p+c_h\right)+c_p Q^\gamma c e^{\theta t_d}\left(1+t_d\theta\right)-\dfrac{N\delta}{\left[1+\delta(T-t_d)\right]^2}\left(\delta c_p-c_b\right)-c_b N_T \\ +c_h Q^\gamma e^{\theta t_d}\left[c(t_d-\dfrac{1}{\theta})\right]+\dfrac{c_h Q^\gamma c}{\theta}\left(2e^{\theta t_d}-1\right) \end{array}\right]$$

$$\frac{\partial^2 TC}{\partial T\partial t_d}=\left[\begin{array}{l} \dfrac{1}{T}\left[\dfrac{\delta N}{\left[1+\delta(T-t_d)\right]^2}(\delta c_p-c_b)+\dfrac{N_T}{\left[1+\delta(T-t_d)\right]}\left(c_b-\delta c_p\right)+c_b N_T\right] \\ -\dfrac{1}{T^2}\left[\begin{array}{l} Q^\gamma (a-bp)\left[e^{\theta t_d}(c_p+\dfrac{c_h}{\theta})-\dfrac{c_h}{\theta}\right]+\dfrac{c_p Q^\gamma c}{\theta}e^{\theta t_d}(1+\theta t_d-1)-\dfrac{N}{\left[1+\delta(T-t_d)\right]}(\delta c_p-c_b) \\ +c_p N_T-Nc_b+c_b N_T\left[T-t_d\right]+\dfrac{c_h Q^\gamma c t_d}{\theta}\left[e^{\theta t_d}-1\right] \end{array}\right] \end{array}\right]$$

$$\frac{\partial^2 TC}{\partial T^2}=\left[\begin{array}{l} \dfrac{1}{T}\left[\begin{array}{l} \dfrac{1}{\left[1+\delta(T-t_d)\right]}\left(Q^\gamma c c_p+\delta c_p N_T-2N_T c_b\right) \\ +\dfrac{N\delta}{\left[1+\delta(T-t_d)\right]^2}(c_b-\delta c_p) \end{array}\right] \\ -\dfrac{2}{T^2}\left[\begin{array}{l} \log\left[1+\delta(T-t_d)\right]\left(c_p N_T-\dfrac{N_T c_b}{\delta}\right)+\dfrac{N}{1+\delta(T-t_d)}\left(\delta c_p-c_b\right)-N_T c_p+Nc_b \\ +c_b N_T\left[t_d-T\right]+Q^\gamma c_a \end{array}\right] \\ +\dfrac{2}{T^3}\left[\begin{array}{l} OC+\dfrac{Q^\gamma(a-bp)}{\theta}\left[e^{\theta t_d}-1\right](c_p+\dfrac{c_h}{\theta})+\dfrac{c_p Q^\gamma c}{\theta}\left[(t_d-\dfrac{1}{\theta})e^{\theta t_d}+\dfrac{1}{\theta})\right] \\ -\dfrac{Q^\gamma c}{\delta}\left[c_p(T-t_d)-c_b\left[Tt_d-\dfrac{{t_d}^2}{2}-\dfrac{T^2}{2}\right]\right]+N\log\left[1+\delta(T-t_d)\right](c_p-\dfrac{c_b}{\delta}) \\ +Nc_b(T-t_d)+\dfrac{c_h Q^\gamma c}{\theta^2}\left[t_d-(t_d-\dfrac{1}{\theta})(1-e^{\theta t_d})\right]-\dfrac{c_h Q^\gamma(a-bp)t_d}{\theta}-\dfrac{c_h Q^\gamma c {t_d}^2}{2\theta}+Q^\gamma c_a T \end{array}\right] \end{array}\right]$$

Appendix B:

Here we proof the theorem mentioned in the paper

$$\alpha=\frac{\partial^2 TC\left(t_2,T\right)}{\partial t_d^2}=\frac{1}{T}\frac{\partial^2 X}{\partial t_d^2}$$

$$=\frac{1}{T}\left[\begin{array}{l} Q^\gamma (a-bp)e^{\theta t_d}\left(\theta c_p+c_h\right)+c_p Q^\gamma c e^{\theta t_d}\left(1+t_d\theta\right) \\ -\dfrac{N\delta}{\left[1+\delta(T-t_d)\right]^2}\left(\delta c_p-c_b\right)-c_b N_T \\ +c_h Q^\gamma e^{\theta t_d}\left[c(t_d-\dfrac{1}{\theta})\right]+\dfrac{c_h Q^\gamma c}{\theta}\left(2e^{\theta t_d}-1\right) \end{array}\right] \tag{B.1}$$

It is clear from the above equation (B.1) that all the terms are positive therefore $\alpha > 0$.

$$\beta = \frac{\partial^2 TC\left(t_2, T\right)}{\partial T^2} = \frac{T^2\left(\frac{\partial^2 X}{\partial T^2} - \frac{\partial X}{\partial T}\right) - 2T\left(T\frac{\partial X}{\partial T} - X\right)}{T^4}$$

$$= \frac{T\left(\frac{\partial^2 X}{\partial T^2} - \frac{\partial X}{\partial T}\right) - 2\left(T\frac{\partial X}{\partial T} - X\right)}{T^3}$$

$$= \frac{T\frac{\partial^2 X}{\partial T^2} - 2T\frac{\partial X}{\partial T} + 2X}{T^3}$$

$$= \left[\begin{array}{l} \frac{1}{T}\left[\begin{array}{l}\frac{1}{[1+\delta(T-t_d)]}\left(Q^\gamma c c_p + \delta c_p N_T - 2N_T c_b\right) \\ + \frac{N\delta}{[1+\delta(T-t_d)]^2}(c_b - \delta c_p)\end{array}\right] \\ -\frac{2}{T^2}\left[\begin{array}{l}\log[1+\delta(T-t_d)]\left(c_p N_T - \frac{N_T c_b}{\delta}\right) + \frac{N}{1+\delta(T-t_d)}\left(\delta c_p - c_b\right) - N_T c_p + N c_b \\ + c_b N_T\left[t_d - T\right] + Q^\gamma c_a\end{array}\right] \\ +\frac{2}{T^3}\left[\begin{array}{l} OC + \frac{Q^\gamma(a-bp)}{\theta}\left[e^{\theta t_d} - 1\right](c_p + \frac{c_h}{\theta}) + \frac{c_p Q^\gamma c}{\theta}\left[(t_d - \frac{1}{\theta})e^{\theta t_d} + \frac{1}{\theta})\right] \\ -\frac{Q^\gamma c}{\delta}\left[c_p(T-t_d) - c_b\left[Tt_d - \frac{t_d^2}{2} - \frac{T^2}{2}\right]\right] + N\log[1+\delta(T-t_d)](c_p - \frac{c_b}{\delta}) \\ + N c_b(T-t_d) + \frac{c_h Q^\gamma c}{\theta^2}\left[t_d - (t_d - \frac{1}{\theta})(1-e^{\theta t_d})\right] - \frac{c_h Q^\gamma(a-bp)t_d}{\theta} - \frac{c_h Q^\gamma c t_d^2}{2\theta} + Q^\gamma c_a T \end{array}\right] \end{array}\right]$$

(B.2)

It is clear from the above equation (B.2) that all the terms are positive therefore $\beta > 0$.

$$\lambda = \frac{\partial^2 TC\left(t_d, T\right)}{\partial T \partial t_d} = -\frac{1}{T^2}\frac{\partial X}{\partial t_d} + \frac{1}{T}\frac{\partial^2 X}{\partial T \partial t_d}$$

$$= \left[\begin{array}{l} \frac{1}{T}\left[\frac{\delta N}{[1+\delta(T-t_d)]^2}(\delta c_p - c_b) + \frac{N_T}{[1+\delta(T-t_d)]}\left(c_b - \delta c_p\right) + c_b N_T\right] \\ -\frac{1}{T^2}\left[\begin{array}{l} Q^\gamma(a-bp)\left[e^{\theta t_d}(c_p + \frac{c_h}{\theta}) - \frac{c_h)}{\theta}\right] + \frac{c_p Q^\gamma c}{\theta}e^{\theta t_d}(1+\theta t_d - 1) - \frac{N}{[1+\delta(T-t_d)]}(\delta c_p - c_b) \\ + c_p N_T - N c_b + c_b N_T\left[T - t_d\right] + \frac{c_h Q^\gamma c t_d}{\theta}\left[e^{\theta t_d} - 1\right] \end{array}\right] \end{array}\right]$$

We know from hessian matrix for two variables:

$$H = \begin{bmatrix} \frac{\partial^2 TC\left(t_d, T\right)}{\partial t_d^2} & \frac{\partial^2 TC\left(t_d, T\right)}{\partial T \partial t_d} \\ \frac{\partial^2 TC\left(t_d, T\right)}{\partial T \partial t_d} & \frac{\partial^2 TC\left(t_d, T\right)}{\partial T^2} \end{bmatrix} = \begin{bmatrix} \alpha & \lambda \\ \lambda & \beta \end{bmatrix} = \alpha\beta - \lambda^2 > 0$$

It's completes the theorem.