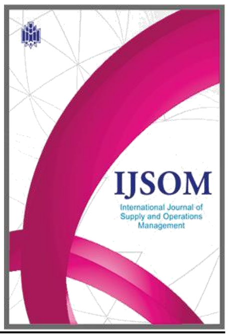
International Journal of Supply and Operations Management

IJSOM

February 2018, Volume 5, Issue 1, pp. 1-10
ISSN-Print: 2383-1359
ISSN-Online: 2383-2525
www.ijsom.com



On the Grey Equal Surplus Sharing Solutions

Uzeyir Alper Yilmaz [a], Sırma Zeynep Alparslan Gok[a,*], Mustafa Ekici [b] and Osman Palanci [a]

[a] *Süleyman Demirel University, Isparta, Turkey*
[b] *Usak University, Usak, Turkey*

Abstract
A situation in which a finite set of players can obtain certain grey payoffs by cooperation can be described by a cooperative grey game. In this paper, we consider some grey division rules, called the grey equal surplus sharing solutions. Further, we focus on a class of the grey equal surplus sharing solutions consisting of all convex combinations of these solutions. Finally, an application of Operations Research (OR) situations is given.

Keywords: Cooperative games; Grey uncertainty; Equal surplus sharing solutions; Facility location situations.

1. Introduction

The grey uncertainty is established by Deng (1982) who introduced a new methodology focusing on the study of problems involving small samples and poor information. It deals with uncertain systems with partially known information through generating, excavating, and extracting useful information from what is available. This paper focuses on some division solutions for cooperative games, called the equal surplus sharing solutions. More precisely, these solutions are Centre-of-gravity of the Imputation-Set value, shortly denoted by CIS-value, Egalitarian Non-Separable Contribution value, shortly denoted by ENSC-value, and the equal division solution, shortly denoted by ED-solution (Driessen and Funaki, 1991; van den Brink and Funaki, 2009). Our main objective in this paper is to extend these solutions by using grey uncertainty.

Cooperative game theory is widely used on interesting cost/profit sharing problems in many areas of OR such as transportation, connection, facility situations, etc. (see Borm et al., 2001 for a survey on OR Games).

In this paper, we present on an application of facility location situations and related games with grey data. Facility location situations are among promising topics in the field of Operations Research (OR), which has many applications to real life. In these types of problems, there exists a given cost for constructing a facility. Further, connecting a player to this facility by minimizing the total cost is necessary (Nisan et al., 2007).

A facility location game is constructed by a facility location situation. We introduce facility location situations with grey data and obtain the facility location grey games. Since these games are defined by using special substraction operator, we apply equal surplus sharing solutions to facility grey situations.

This paper is organized as follows. Section 2 considers some preliminaries on cooperative game theory, cooperative grey games, grey calculus, and classical equal surplus sharing solutions in cooperative game theory. In Section 3, we introduce grey game-theoretical equal surplus sharing solutions on special subclass of cooperative grey games. Section 4 discusses a class of equal surplus sharing grey solutions. In Section 5, we give an application of facility location situations with grey data.

Corresponding author email address: zeynepalparslan@yahoo.com

2. Preliminaries

In this section, some preliminaries about cooperative games and grey calculus are given (Alparslan Gok et al., 2009a,2009b; Branzei et al., 2008; Deng, 1982; Palanci et al., 2015; Tijs, 2003).

A *cooperative game* in coalitional form is an ordered pair $< N, v >$, where $N = \{1,2,\ldots,n\}$ is the set of players, and $v : 2^N \to \mathrm{P}$ is a map, assigning to each coalition $S \in 2^N$ a real number $v(S)$, such that $v(\emptyset) = 0.$

We identify a cooperative game $< N, v >$ with its characteristic function $v.$ The family of all games with player set N is denoted by $G^N.$

In this paper, we consider a one point solution f on G^N assings that a payoff vector $f(v) \in \mathrm{P}^N$ to every cooperative game $v \in G^N.$ Examples of such solutions are the Centre-of-gravity of the Imputation-Set value, shortly denoted by CIS-value , Egalitarian Non-Separable Contribution value, shortly denoted by ENSC-value, and the equal division solution (Driessen and Funaki, 1991; van den Brink and Funaki, 2009).

The CIS-value, the ENSC-value, and the ED-solution are defined by as follows:

$$CIS_i(v) = v(\{i\}) + \frac{1}{|N|}(v(N) - \sum_{j \in N} v(\{j\})) \tag{1}$$

for all $i \in N$.

$$ENSC_i(v) = -v(N \setminus \{i\}) + \frac{1}{|N|}(v(N) + \sum_{j \in N} v(N \setminus \{j\})) \text{ forall} i \in N. \tag{2}$$

for all $i \in N$.

$$ED_i(v) = \frac{1}{|N|} v(N) \tag{3}$$

for all $i \in N$.

A grey number that a number whose exact value is known but a range within that the value lies is known. In applications, a grey number in general is an interval or a general set of numbers. In this paper, we consider the interval grey numbers. A grey number with both a lower limit $\underline{a}$ and a upper limit $\overline{a}$ is called an interval grey number, denoted as $w' \in [\underline{a}, \overline{a}]$. For example, the weight of a whale is between 150 and 200 ton. The length of a giraffe is between 4 and 5 meters. These two grey numbers can be respectively written as

$w_1' \in [150,200]$ and $w_2' \in [4,5]$.

Now, we discuss various operations on interval grey numbers.

Let

$w_1' \in [a,b], a < b$ and $w_2' \in [c,d], c < d.$

The sum of w_1' and w_2', written $w_1' + w_2'$, is defined as follows:

$w_1' + w_2' \in [a + c, b + d]$.

For example, let $w_1' \in [3,4]$ and $w_2' \in [5,8]$, then,

$w_1' + w_2' \in [8,12]$.

Assume that $w' \in [a,b]$, $a < b,$ and k is a positive real number. The scalar multiplication of k and w' is defined as follows:

$kw' \in [ka, kb]$.

We denote by $\Gamma(\mathrm{P})$ the set of interval grey numbers in P.

Let $w_1', w_2' \in \Gamma(\mathrm{P})$ with $w_1' \in [a,b]$, $a < b$; $w_2' \in [c,d]$, $c < d$, $|w_1'| = b - a$ and $\alpha \in \mathrm{P}_+$. Then,

(i) $w_1' + w_2' \in [a + c, b + d]$;

(ii) $\alpha w' \in [\alpha a, \alpha b]$.

By (i) and (ii) we see that $\Gamma(\mathrm{P})$ has a cone structure.

In general, the difference of $w_1^{'}$ and $w_2^{'}$ is defined as follows:

$$w_1^{'} ! \ w_2^{'} = w_1^{'} + \left(-w_2^{'}\right) \in [a-d, b-c],$$

(see Moore, 1979).

For example, let $w_1^{'} \in [6,8]$, and $w_2^{'} \in [2,5]$, then we have

$$w_1^{'} ! \ w_2^{'} \in [6-5, 8-2] = [1,6],$$

$$w_2^{'} ! \ w_1^{'} \in [2-8, 5-6] = [-6,-1].$$

Different from the above subtraction, we use a partial subtraction operator (for details see Alparslan Gok et al., 2009a,2011; Branzei et al., 2008). We define $w_1^{'} - w_2^{'}$, only if $|b-a| \geq |d-c|$, by $w_1^{'} - w_2^{'} \in [a-c, b-d]$ (for details see Alparslan Gok et al., 2009a; Branzei et al., 2008).

Notice that if we make a comparison with the above example, then in our case $[3,6]-[4,8]$ is not defined. But, $[4,8]-[3,6]$ is defined.

Let $w_1^{'} \in [4,8]$, and $w_2^{'} \in [3,6]$, $w_1^{'} - w_2^{'}$ is defined since $|8-4| \geq |6-3|$, but $w_2^{'} - w_1^{'}$ is not defined since $|6-3| = 3 \square 4 = |8-4|$, then we have

$$w_1^{'} - w_2^{'} \in [4-3, 8-6] = [1,2].$$

A cooperative grey game is an ordered pair $< N, w^{'} >$, where $N = \{1, \ldots, n\}$ is the set of players, and $w^{'} : 2^N \to \Gamma(\mathrm{P})$ is the characteristic function such that $w^{'}(\emptyset) \in [0,0]$, grey payoff function $w^{'}(S) \in [\underline{A_S}, \overline{A_S}]$ refers to the value of the grey expectation benefit belonging to a coalition $S \in 2^N$, where $\overline{A_S}$ and $\underline{A_S}$ represent possible maximum and minimum profits of the coalition S. So, a cooperative grey game can be considered as a classical cooperative game with grey profits $w^{'}$. Grey solutions are useful to solve profit/cost sharing problems with grey data using cooperative grey games as a tool. Building blocks for grey solutions are grey payoff vectors, i.e., vectors whose components belong to $\Gamma(\mathrm{P})$. We denote by $\Gamma(\mathrm{P})^N$ the set of all such grey payoff vectors. We denote by ΓG^N the family of all cooperative grey games.

Now, we introduce some theoretical notions from the theory of cooperative grey games.

For $w, w_1, w_2 \in IG^N$ and $w_1^{'}, w_2^{'} \in \Gamma G^N$ we say that $w_1^{'} \in w_1 \leq w_2^{'} \in w_2$ if $w_1^{'}(S) \leq w_2^{'}(S)$, where $w_1^{'}(S) \in w_1(S)$ and $w_2^{'}(S) \in w_2(S)$ and, for each $S \in 2^N$.

For $w_1^{'}, w_2^{'} \in \Gamma G^N$ and $\lambda \in \mathrm{P}_+$ we define $< N, w_1^{'} + w_2^{'} >$ and $< N, \lambda w^{'} >$

by $(w_1^{'} + w_2^{'})(S) = w_1^{'}(S) + w_2^{'}(S)$ and $(\lambda w^{'})(S) = \lambda w^{'}(S)$ for each $S \in 2^N$. So, we conclude that ΓG^N endowed with "$\leq$" has a cone structure with respect to addition and multiplication with non-negative scalars above. For $w_1^{'}, w_2^{'} \in \Gamma G^N$ where $w_1^{'} \in w_1, w_2^{'} \in w_2$ with $|w_1(S)| \geq |w_2(S)|$ for each $S \in 2^N$, $< N, w_1^{'} - w_2^{'} >$ is defined by $(w_1^{'} - w_2^{'})(S) = w_1^{'}(S) - w_2^{'}(S) \in w_1(S) - w_2(S)$.

We call a game $< N, w^{'} >$ *grey size monotonic* if $< N, |w| >$ is monotonic, i.e., $|w|(S) \leq |w|(T)$ for all $S, T \in 2^N$ with $S \subset T$. For further use we denote by $SM\Gamma G^N$ the class of grey size monotonic games with player set N.

The grey CIS-value and the grey $ENSC$-value are defined on $SM\Gamma G^N$ but the grey ED-solution is only defined on ΓG^N.

3. The grey equal surplus sharing solutions

In this section, we introduce some game-theoretic solutions by using grey calculus which are inspired by van den Brink and Funaki (2009).

The grey CIS -value (ΓCIS -value) assigns every player to its individual grey worth, and distributes the remainder of the grey worth of the grand coalition N, equally among all players.

The ΓCIS -value is defined by

$$\Gamma CIS : SM\Gamma G^N \to \Gamma(\mathrm{P})^N \tag{4}$$
$$\Gamma CIS_i\left(w'\right) = w'(\{i\}) + \tfrac{1}{|N|}\left(w'(N) - \sum\nolimits_{j\in N} w'(\{j\})\right)$$

such that

$$\left|w'(N)\right| \geq \sum\nolimits_{j\in N}\left|w'(\{j\})\right|$$

for all $i \in N$ and for all $w' \in SM\Gamma G^N$.

Example 3.1 *Let* $w' \in SM\Gamma G^N$ *and* $N = \{1,2,3\}$. *The coalitional values are as follows:*

$$w'(\varnothing) = w'(3) = [0,0],$$
$$w'(1) = w'(2) = [1,2],$$
$$w'(12) = [4,7], w'(13) = [1,4],$$
$$w'(23) = [3,5] \text{ and } w(N) = [5,9].$$

The ΓCIS -value of the game is calculated as follows:

$$\begin{aligned}
&\Gamma CIS_1\left(w'\right) \in w'(1) + \tfrac{1}{3}\left(w'(N) - \left(w'(1) + w'(2) + w'(3)\right)\right)\\
&= [1,2] + \tfrac{1}{3}\left([5,9] - [2,4]\right)\\
&= [1,2] + \tfrac{1}{3}[3,5]\\
&= \left[2,3\tfrac{2}{3}\right],\\
&\Gamma CIS_2\left(w'\right) \in w'(2) + \tfrac{1}{3}\left(w'(N) - \left(w'(1) + w'(2) + w'(3)\right)\right)\\
&= [1,2] + \tfrac{1}{3}[3,5]\\
&= \left[2,3\tfrac{2}{3}\right],\\
&\Gamma CIS_3\left(w'\right) \in w'(3) + \tfrac{1}{3}\left(w'(N) - \left(w'(1) + w'(2) + w'(3)\right)\right)\\
&= [0,0] + \tfrac{1}{3}[3,5]\\
&= \left[1,1\tfrac{2}{3}\right].
\end{aligned}$$

Then, the ΓCIS -value is

$$\Gamma CSI\left(w'\right) \in \left(\left[2,3\tfrac{2}{3}\right], \left[2,3\tfrac{2}{3}\right], \left[1,1\tfrac{2}{3}\right]\right).$$

The dual $w'^* \in SM\Gamma G^N$ of the grey game w' is the game that assigns to each coalition $S \subseteq N$ the grey worth that is lost by the grand coalition N if coalition S leaves N, i.e.

$$w'^*(S) = w'(N) - w'(N \setminus S)$$

for all $S \subseteq N$.

The grey $ENSC$ -value ($\Gamma ENSC$ -value) assigns to every game w' the ΓCIS -value of its dual game, i.e.

$$\Gamma ENSC : SM\Gamma G^N \to \Gamma(\mathrm{P})^N \tag{5}$$
$$\begin{aligned}
&\Gamma ENSC_i\left(w'\right) = \Gamma CIS_i\left(w'^*\right)\\
&= \tfrac{1}{|N|}\left(w'(N) + \sum\nolimits_{j\in N} w'(N \setminus \{j\})\right) - w'(N \setminus \{i\})
\end{aligned}$$

such that

$$\left|w'(N)+\sum_{j\in N} w'(N\setminus\{j\})\right| \geq |N|\left|w'(N\setminus\{i\})\right|$$

for all $i \in N$ and for all $w' \in SM\Gamma G^N$.

Thus, the $\Gamma ENSC$-value assigns to every player in a game its grey marginal contribution to the "grand coalition" and distributes the remainder equally among the players.

Example 3.2 *Consider the game in Example 3.1. We calculate the $\Gamma ENSC$-value of the game in Example 3.1 as follows:*

$$\begin{aligned}
&\Gamma ENSC_1(w') \in \tfrac{1}{3}(w'(N)+w'(23)+w'(12)+w'(13)) - w'(23)\\
&= \tfrac{1}{3}([13,25]) - [3,5]\\
&= \left[1\tfrac{1}{3},3\tfrac{1}{3}\right],\\
&\Gamma ENSC_2(w') \in \tfrac{1}{3}(w'(N)+w'(23)+w'(12)+w'(13)) - w'(13)\\
&= \tfrac{1}{3}([13,25]) - [1,4]\\
&= \left[3\tfrac{1}{3},4\tfrac{1}{3}\right],\\
&\Gamma ENSC_3(w') \in \tfrac{1}{3}(w'(N)+w'(23)+w'(12)+w'(13)) - w'(12)\\
&= \tfrac{1}{3}([13,25]) - [4,7]\\
&= \left[\tfrac{1}{3},1\tfrac{1}{3}\right].
\end{aligned}$$

Then, the $\Gamma ENSC$-value is given

$$IENSC(w') = \left(\left[1\tfrac{1}{3},3\tfrac{1}{3}\right],\left[3\tfrac{1}{3},4\tfrac{1}{3}\right],\left[\tfrac{1}{3},1\tfrac{1}{3}\right]\right).$$

The $\Gamma ENCIS$-value is defined for $\beta \in [0,1]$ as follows:

$$\begin{aligned}
&\Gamma ENCIS : SM\Gamma G^N \to \Gamma(\mathrm{P})^N \qquad (6)\\
&\Gamma ENCIS^{\beta}(w') = \beta\Gamma CIS(w') + (1-\beta)\Gamma ENSC(w')
\end{aligned}$$

such that

$$|w'|(i) = |w'|(N\setminus\{i\}) = |w'|(j)$$

for all $i, j \in N$ with $i \neq j$.

Finally, the grey ED-solution (ΓED-solution) is given by

$$\begin{aligned}
&\Gamma ED : \Gamma G^N \to \Gamma(\mathrm{P})^N \qquad (7)\\
&\Gamma ED_i(w') = \frac{w'(N)}{|N|}
\end{aligned}$$

for all $i \in N$

Example 3.3 *Consider the game in Example 3.1. The ΓED-solution of the game in Example 3.1 is calculated as follows:*

$$\Gamma ED_1(w') = \Gamma ED_2(w') = \Gamma ED_3(w') \in \frac{w'(N)}{3} = \frac{[5,9]}{3} = \left[1\tfrac{2}{3},3\right]$$

Then, the ΓED-solution is obtained by

$$\Gamma ED(w') \in \left(\left[1\tfrac{2}{3},3\right],\left[1\tfrac{2}{3},3\right],\left[1\tfrac{2}{3},3\right]\right).$$

Remark 3.1 *We note that the ΓCIS-value and the $\Gamma ENSC$-value are defined in $SM\Gamma G^N$, but the ΓED-solution is defined in ΓG^N.*

4. A class of grey equal surplus sharing solutions

In this paper, we discuss a class of grey solutions that consists of all convex combinations of the ΓED-solution, the ΓCIS -value, and the $\Gamma ENSC$ -value, i.e., for $\alpha, \beta \in [0,1]$, we consider grey solutions $\Gamma\phi^{\alpha,\beta}$ given by

$$\Gamma\phi^{\alpha,\beta}\left(w^{'}\right) = \alpha\Gamma ENCIS^{\beta}\left(w^{'}\right) + (1-\alpha)\Gamma ED\left(w^{'}\right), \tag{8}$$

where $\Gamma ENCIS^{\beta}\left(w^{'}\right)$ is given by (1). We denote the class of all grey solutions that are obtained in this way by $\Gamma\Phi := \left\{\Gamma\phi^{\alpha,\beta} : \alpha, \beta \in [0,1]\right\}$. Clearly, the interesting solutions in this class are the ΓCIS -value, which is obtained by taking $\alpha = \beta = 1$ (i.e. $\Gamma CIS\left(w^{'}\right) = \phi^{1,1}\left(w^{'}\right)$), the $\Gamma ENSC$ -value, which is obtained by taking $\alpha = 1,\ \beta = 0$ (i.e. $\Gamma ENSC\left(w^{'}\right) = \Gamma\phi^{1,0}\left(w^{'}\right)$) and the ΓED-solution, which is obtained by taking $\alpha = 0$ (i.e. $\Gamma ED\left(w^{'}\right) = \Gamma\phi^{0,\beta}$, $\beta \in [0,1]$). We thus can write $\Gamma\phi^{\alpha,\beta}$ as

$$\begin{aligned}\Gamma\phi^{\alpha,\beta}\left(w^{'}\right) &= \alpha\Gamma\phi^{1,\beta}\left(w^{'}\right) + \left(1-w^{'}\right)\Gamma\phi^{0,1}\left(w^{'}\right) \\ &= \alpha\beta\Gamma\phi^{1,1}\left(w^{'}\right) + \alpha(1-\beta)\Gamma\phi^{1,0}\left(w^{'}\right) + (1-\alpha)\Gamma\phi^{0,1}\left(w^{'}\right)\end{aligned} \tag{9}$$

for $\alpha, \beta \in [0,1]$.

Proposition 4.1 gives an expression of the solutions $\Gamma\phi^{\alpha,\beta}$ in the sense that they give each player i in a grey game $w^{'}$ some value $\lambda_i^{\alpha,\beta}\left(w^{'}\right)$, and the remainder of $w^{'}(N)$ is equally divided among all players.

Proposition 4.1 *For every* $w^{'} \in SM\Gamma G^N$ *and* $\alpha, \beta \in [0,1]$ *it holds that*

$$\Gamma\phi_i^{\alpha,\beta}\left(w^{'}\right) = \lambda_i^{\alpha,\beta}\left(w^{'}\right) + \tfrac{1}{|N|}\left(w^{'}(N) - \sum\nolimits_{j\in N}\lambda_j^{\alpha,\beta}\left(w^{'}\right)\right), \tag{10}$$

wher $\lambda_i^{\alpha,\beta}\left(w^{'}\right) = \alpha\left(\beta w^{'}(\{i\}) - (1-\beta)w^{'}(N\setminus\{i\})\right)$ for $i \in N$ such that $\left|w^{'}\right|(i) = \left|w^{'}\right|(N\setminus\{i\}) = \left|w^{'}\right|(j)$ for all $i, j \in N$ with $i \neq j$.

Proof.[1] For $w^{'} \in SM\Gamma G^N$ and $\alpha, \beta \in [0,1]$ we have

$$\begin{aligned}\Gamma\phi_i^{\alpha,\beta}\left(w^{'}\right) &\in \alpha\Gamma ENCIS^{\beta}\left(w^{'}\right) + (1-\alpha)\Gamma ED\left(w^{'}\right) \\ &= \alpha\left(\beta w^{'}(\{i\}) - (1-\beta)w^{'}(N\setminus\{i\})\right) \\ &+ \frac{\alpha}{|N|}\left(w^{'}(N) - \sum_{j\in N}\left(\beta w^{'}(j) - (1-\beta)w^{'}(N\setminus\{j\})\right)\right) + \frac{1-\alpha}{|N|}w^{'}(N) \\ &= \alpha\left(\beta w^{'}(\{i\}) - (1-\beta)w^{'}(N\setminus\{i\})\right) \\ &+ \frac{1}{|N|}\left(w^{'}(N) - \sum_{j\in N}\alpha\left(\beta w^{'}(j) - (1-\beta)w^{'}(N\setminus\{j\})\right)\right) \\ &= \lambda_i^{\alpha,\beta}\left(w^{'}\right) + \tfrac{1}{|N|}\left(w^{'}(N) - \sum\nolimits_{j\in N}\lambda_j^{\alpha,\beta}\left(w^{'}\right)\right).\end{aligned}$$

Proposition 4.2 *For every* $\alpha, \beta \in [0,1]$ *and* $w^{'} \in SM\Gamma G^N$ *it holds that* $\Gamma\phi_i^{\alpha,\beta}\left(w^{'*}\right) = \Gamma\phi^{\alpha,1-\beta}\left(w^{'}\right)$.

Proof.[2]. For $w^{'} \in SM\Gamma G^N$ and $\alpha, \beta \in [0,1]$ we have

$$\begin{aligned}\Gamma\phi_i^{\alpha,\beta}\left(w^{'*}\right) &= \lambda_i^{\alpha,\beta}\left(w^{'*}\right) + \tfrac{1}{|N|}\left(w^{'*}(N) - \sum\nolimits_{j\in N}\lambda_j^{\alpha,\beta}\left(w^{'*}\right)\right) \\ &= \alpha\left(\beta w^{'*}(\{i\}) - (1-\beta)w^{'*}(N\setminus\{i\})\right) \\ &+ \frac{1}{|N|}\left(w^{'*}(N) - \sum_{j\in N}\alpha\left(\beta w^{'*}(j) - (1-\beta)w^{'*}(N\setminus\{j\})\right)\right)\end{aligned}$$

---

[1] In this proof, we inspired by van den Brink and Funaki (2009).
[2] In this proof, we inspired by van den Brink and Funaki (2009).

$$
\begin{aligned}
&= \alpha(\beta w'(N)) - \beta w'(N \setminus \{i\}) - (1-\beta) w'(N) + (1-\beta) w'(\{i\})) \\
&+ \frac{1}{|N|}\left(w'(N) - \sum_{j \in N} \alpha(\beta w'(N) - \beta w'(N \setminus \{j\}) - (1-\beta) w'(N) + (1-\beta) w'(j) \right) \\
&= w'(N)(\alpha\beta - \alpha(1-\beta) + \frac{1}{|N|} - \frac{|N|\alpha\beta}{|N|} + \frac{|N|\alpha(1-\beta)}{|N|}) \\
&+ \alpha\left((1-\beta) w'(i) - \beta\left(w'(N \setminus \{i\})\right) + \frac{1}{|N|} \sum_{j \in N} \left(\beta w'(N \setminus \{j\}) - (1-\beta) w'(j)\right) \right) \\
&= \frac{1}{|N|} w'(N) + \alpha\left((1-\beta) w'(\{i\}) - \beta w'(N \setminus \{i\})\right) \\
&- \frac{1}{|N|}\left(\sum_{j \in N} \alpha\left((1-\beta) w'(\{j\}) - \beta w'(N \setminus \{j\})\right) \right) \\
&= \Gamma\phi^{\alpha, 1-\beta}\left(w'\right).
\end{aligned}
$$

5. An application

In a facility location game, a set A of agents (also known as cities, clients, or demand points), a set Φ of facilities, a facility opening cost f_i for every facility $i \in \Phi$, and a distance d_{ij} between every pair (i, j) of points in $\mathrm{A} \cup \Phi$ indicating the cost of connecting j to i are given. We assume that the distances come from a metric space; i.e., they are symmetric and obey the triangle inequality. For a set $S \subseteq \mathrm{A}$ of agents, the cost of this set is defined as the minimum cost of opening a set of facilities and connecting every agent in S to an open facility. More precisely, the cost function c is defined by

$$
c(S) = \min_{\Phi^* \subseteq \Phi} \{ \sum_{i \in \Phi^*} f_i + \sum_{j \in S} \min_{i \in \Phi^*} d_{ij} \}. \tag{11}
$$

Facility location games are studied by Nisan et al. (2007) in the literature. Further, facility location interval games are introduced by Palanci et al. (2017).

In this study, we introduce the facility location grey games. In a facility location grey game, a set A of agents (also known as cities, clients, or demand points), a set Φ of facilities, a grey facility opening cost f_i' for every facility $i \in \Phi$, and a distance d_{ij}' between every pair (i, j) of points in $\mathrm{A} \cup \Phi$ indicating the rey cost of connecting j to i are given. Here, $f_i' \in \left[\underline{f_i'}, \overline{f_i'}\right], d_{ij}' \in \left[\underline{d_{ij}'}, \overline{d_{ij}'}\right] \in \Gamma(\mathrm{P})$. The distances are supposed to come from a metric space. So, these distances are symmetric and satisfy the triangle inequality. For a set $S \subseteq \mathrm{A}$ of agents, the grey cost of this set is defined as the minimum grey cost of opening a set of facilities and connecting every agent in S to an open facility. More precisely, the grey cost function w' is defined by

$$
w'(S) \in \left[\min_{\Phi^* \subseteq \Phi} \{ \sum_{i \in \Phi^*} \underline{f_i} + \sum_{j \in S} \min_{i \in \Phi^*} \underline{d_{ij}} \}, \min_{\Phi^* \subseteq \Phi} \{ \sum_{i \in \Phi^*} \overline{f_i} + \sum_{j \in S} \min_{i \in \Phi^*} \overline{d_{ij}} \} \right] \in \Gamma(\mathrm{P}) \tag{12}
$$

Now, we present an application from a facility location situation and related game with grey data.

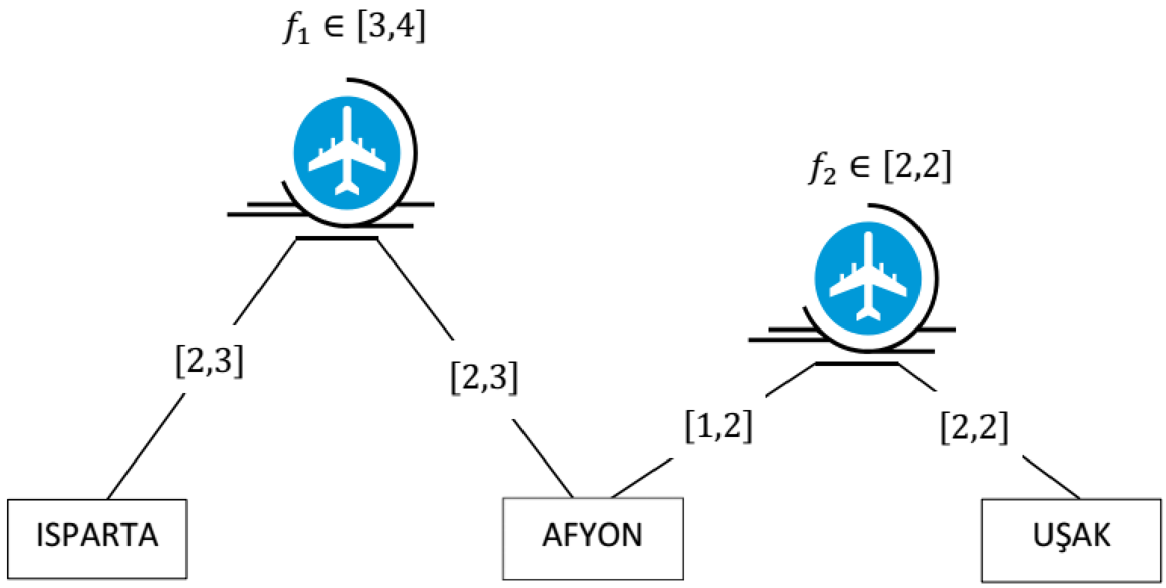


Figure 1. An application of a facility location situation from Turkey.

Example 5.1 *Figure 1 shows a facility location grey situation with 3 cities {Isparta (Player 1), Afyon (Player 2), Usak (Player 3)} and 2 airports* $\{1,2\}$ *in Turkey. The grey cost function is calculated as follows:*

$$w'(1)\in[5,7], w'(2)\in[3,4], w'(3)\in[4,4],$$
$$w'(12)\in[7,10], w'(13)\in[9,11], w'(23)\in[5,6],$$
$$w'(123)\in[10,13].$$

Now, we want to calculate ΓCIS-value, $\Gamma ENSC$-value and ΓED-solution.We calculate the ΓCIS-value of this game as follows:

$$\Gamma CIS_1(w')\in w'(1)+\tfrac{1}{3}(w'(N)-(w'(1)+w'(2)+w'(3)))$$
$$=[4\tfrac{1}{3},6\tfrac{1}{3}]$$
$$\Gamma CIS_2(w')\in w'(2)+\tfrac{1}{3}(w'(N)-(w'(1)+w'(2)+w'(3)))$$
$$=[2\tfrac{1}{3},3\tfrac{1}{3}],$$
$$\Gamma CIS_3(w')\in w'(3)+\tfrac{1}{3}(w'(N)-(w'(1)+w'(2)+w'(3)))$$
$$=[3\tfrac{1}{3},3\tfrac{1}{3}],$$

Then, the ΓCIS-value is obtained by

$$\Gamma CIS(w')\in([4\tfrac{1}{3},6\tfrac{1}{3}],[2\tfrac{1}{3},3\tfrac{1}{3}],[3\tfrac{1}{3},3\tfrac{1}{3}]).$$

We calculate the $\Gamma ENSC$-value of this game as follows:

$$\Gamma ENSC_1(w')\in -w'(23)+\tfrac{1}{3}(w'(N)+w'(23)+w'(12)+w'(13))$$
$$=[5\tfrac{1}{3},7\tfrac{1}{3}],$$
$$\Gamma ENSC_2(w')\in -w'(13)+\tfrac{1}{3}(w'(N)+w'(23)+w'(12)+w'(13))$$
$$=[1\tfrac{1}{3},2\tfrac{1}{3}],$$
$$\Gamma ENSC_3(w')\in -w'(12)+\tfrac{1}{3}(w'(N)+w'(23)+w'(12)+w'(13))$$
$$=-[6,20]+\tfrac{1}{3}([22,66])$$
$$=[3\tfrac{1}{3},3\tfrac{1}{3}].$$

Then, the $\Gamma ENSC$ -value is obtained by

$$\Gamma ENSC(w') = ([5\tfrac{1}{3}, 7\tfrac{1}{3}], [1\tfrac{1}{3}, 2\tfrac{1}{3}], [3\tfrac{1}{3}, 3\tfrac{1}{3}]).$$

Finally, we calculate the ΓED-solution of this game as follows:

$$\Gamma ED_1(w') = \Gamma ED_2(w') = \Gamma ED_3(w') \in \frac{w'(N)}{3} = \frac{[10,13]}{3} = [3\tfrac{1}{3}, 4\tfrac{1}{3}].$$

Then, the ΓED-solution is obtained by

$$\Gamma ED(w') = ([3\tfrac{1}{3}, 4\tfrac{1}{3}], [3\tfrac{1}{3}, 4\tfrac{1}{3}], [3\tfrac{1}{3}, 4\tfrac{1}{3}]).$$

Table 1[3] illustrates the results of this application.

Table 1. The grey equal surplus sharing solutions of Example 5.1.

Grey Solutions	Player 1	Player 2	Player 3
ΓCIS -value	$[4\tfrac{1}{3}, 6\tfrac{1}{3}]$	$[2\tfrac{1}{3}, 3\tfrac{1}{3}]$	$[3\tfrac{1}{3}, 3\tfrac{1}{3}]$
$\Gamma ENSC$ -value	$[5\tfrac{1}{3}, 7\tfrac{1}{3}]$	$[1\tfrac{1}{3}, 2\tfrac{1}{3}]$	$[3\tfrac{1}{3}, 3\tfrac{1}{3}]$
ΓED-solution	$[3\tfrac{1}{3}, 4\tfrac{1}{3}]$	$[3\tfrac{1}{3}, 4\tfrac{1}{3}]$	$[3\tfrac{1}{3}, 4\tfrac{1}{3}]$

6. Conclusion

In this paper we propose different equal surplus sharing solutions by using grey numbers. Uncertainty with grey numbers is the natural type of uncertainty which may influence cooperation because lower and upper bounds for future outcomes or costs of cooperation can always be estimated based on available economic data. Moreover, we give an application on facility location grey games.

As a future work, the class of cooperative grey games can be applied to different economic and Operations Research (OR) problems such as bankruptcy situations, airport situations, sequencing situations and cost allocation problems arising from connection situations. We note that the obtained results can be extended to different application areas such as OR and economic situations with grey numbers. It would be interesting to extend other solution concepts by using grey calculus such as Banzhaf value, τ -value etc.

References

Alparslan Gok S.Z., Branzei R. and Tijs S., (2009a). Convex interval games. *Journal of Applied Mathematics and Decision Sciences*, 2009, Article ID 342089, 14 pages, DOI: 10.1115/2009/342089.

Alparslan Gok S.Z., Miquel S. and Tijs S., (2009b). Cooperation under interval uncertainty. *Mathematical Methods of Operations Research*, Vol. 69, pp. 99-109.

Alparslan Gok S.Z., Branzei O., Branzei R. and Tijs S., (2011). Set-valued solution concepts using interval-type payoffs for interval games, *Journal of Mathematical Economics*, Vol.47, pp. 621-626.

Borm P., Hamers H. and Hendrickx R., (2001). Operations Research games: A survey. *TOP*, Vol.9, pp. 139-216.

Branzei R., Dimitrov D. and Tijs S., (2008). *Models in Cooperative Game Theory*. Game Theory and Mathematical Methods, Springer, Berlin, Germany.

Deng J., (1982). Control problems of Grey Systems, *Systems and Control Letters*, Vol.5, pp. 288-294.

Driessen T.S.H. and Funaki Y., (1991). Coincidence of and collinearity between game theoretic solutions. *OR Spektrum*, Vol. 13, pp. 15-30.

Moore R., (1979). *Methods and applications of interval analysis*, SIAM, Philadelphia.

[3] The results of this application are calculated by MAPLE software programme.

Nisan N., Roughgarden T., Tardos É. and Vazirani V.V., (2007). Algorithmic Game Theory, Cambridge, UK: Cambridge University Press, ISBN 0-521-87282-0.

Palanci O., Alparslan Gok S.Z., Ergun S. and Weber G.-W., (2015). Cooperative grey games and the grey Shapley value, *Optimization: A Journal of Mathematical Programming and Operations Research*, Vol. 64(8), pp. 1657-1668.

Palanci O. and Alparslan Gok S.Z., (2017). Facility Location Situations and Related Games in Cooperation. Spatial Interaction Models, Vol. 118 (The series Springer Optimization and Its Applications), pp. 247-260.

Tijs S., (2003). *Introduction to Game Theory*, Hindustan Book Agency, India.

van den Brink R. and Funaki Y., (2009). Axiomatizations of a Class of Equal Surplus Sharing Solutions for TU-Games, *Theory and Decision*, Vol. 67(3), pp. 303-340.